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Numerical Simulation (V4E2)
Summer semester 2012
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Problem Sheet 3

1. **Energy norm stability of a hierarchical FE basis**
Let $D = (0,1)$, $V = H_0^1(D)$ and

$$\phi(x) = \begin{cases} 2x & 0 < x \le \frac{1}{2}, \\ 1 - 2x & \frac{1}{2} < x < 1, \\ 0 & \text{else.} \end{cases} \tag{1}$$

We want to approximate V by the hierarchical Finite Element spaces

$$V_L := \operatorname{span}\{\psi_j^\ell,\ 0 \le \ell \le L,\ 1 \le j \le 2^\ell\}, \tag{2}$$

where

$$\psi_j^\ell(x) := 2^{-\frac{\ell}{2}} \phi\left(2^\ell x - j + 1\right). \tag{3}$$

Any element $v_L \in V_L$ can be written as

$$v_L = \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} v_j^\ell \psi_j^\ell, \tag{4}$$

where $M_\ell = 2^\ell$ and $v_j^\ell \in \mathbb{R}$.
Prove that there exists a constant C_B independent of L, sucht that for every $v_L \in V_L$

$$C_B^{-1} \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} |v_j^\ell|^2 \le \|v_L\|_V^2 \le C_B \sum_{\ell=0}^{L} \sum_{j=1}^{M_\ell} |v_j^\ell|^2 \tag{5}$$

holds. (Show that Example 3.5 fullfills Assumption 3.4 from the lecture.)

2. Memory requirements for realization of a sparse-tensor matrix-vector multiplication

Let $A^L_{\ell,\ell'} = \left\langle A\Psi^\ell, \Psi^{\ell'} \right\rangle$, where A is an elliptic operator, $\Psi^\ell = \left\{\psi^\ell_j, 1 \le j \le M_\ell\right\}$, ψ^ℓ_j the basis functions of W_ℓ and $M_\ell = \dim(W_\ell) \sim 2^{\ell d}$. Assume that

$$\#\,\mathrm{nnz}(A^L_{\ell,\ell'}) \le C_d(\min(\ell,\ell') + 1)^{d-1} 2^{d \max(\ell,\ell')}.$$

Prove that Algorithm 3.8 from the lecture (below) requieres a storage of order $\mathcal{O}(L^d 2^{dL})$.

Store $A^L_{\ell,\ell'}$, $\ell, \ell' = 0, \dots, L$ and $(c_{\ell_1'\ell_2'})_{\ell_1'+\ell_2' \le L}$
for ℓ_1, ℓ_2: $\ell_1 + \ell_2 \le L$ **do**
  $x_{\ell_1\ell_2} := 0$
  for ℓ_1', ℓ_2': $\ell_1' + \ell_2' \le L$ **do**
    if $(\ell_1 + \ell_2' \le \ell_1' + \ell_2)$ **then**
      $y_{\ell_1\ell_2} = (A^L_{\ell_1\ell_1'} c_{\ell_1'\ell_2'})(A^L_{\ell_2\ell_2'})^T$
    else
      $y_{\ell_1\ell_2} = A^L_{\ell_1\ell_1'}(c_{\ell_1'\ell_2'}(A^L_{\ell_2\ell_2'})^T)$
    end if
    $x_{\ell_1\ell_2} = x_{\ell_1\ell_2} + y_{\ell_1\ell_2}$
  end for
end for
for ℓ_1, ℓ_2: $\ell_1 + \ell_2 \le L$ **do**
  $c_{\ell_1\ell_2} = x_{\ell_1\ell_2}$
end for

Date of submission: Friday, 4 May 2012
Website: `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`