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Numerical Simulation (V4E2)
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Problem Sheet 5

1. **Convergence rate for the full tensor product spectral discretization of a k-th moment equation**

Let $u \in H^s_{\text{iso}}(D^k)$ for some speroid D and a real number $s \geq \frac{1}{2}$,where

$$H^s_{\text{iso}}(D^k) := H^s(D) \otimes H^{\frac{1}{2}}(D) \otimes \ldots \otimes H^{\frac{1}{2}}(D) \cap \ldots \cap H^{\frac{1}{2}}(D) \otimes \ldots H^{\frac{1}{2}}(D) \otimes H^s(D) \tag{1}$$

We can write

$$u = \sum_{\underline{\ell}=0}^{\infty} \sum_{\underline{m}=-\underline{\ell}}^{\underline{\ell}} \hat{u}_{\underline{\ell},\underline{m}} Y_{\underline{\ell},\underline{m}}. \tag{2}$$

We have defined the full tensor product space

$$S_L := \text{span}\left\{ Y_{\underline{\ell},\underline{m}} : \ 0 \leq \ell_i \leq L \ ; \ |m_i| \leq \ell_i \right\} \tag{3}$$

and by the orthogonality of $Y_{\underline{\ell},\underline{m}}$ the projection from L^2 to S_L

$$P_L u = \sum_{\underline{\ell}=0}^{L} \sum_{\underline{m}=-\underline{\ell}}^{\underline{\ell}} \hat{u}_{\underline{\ell},\underline{m}} Y_{\underline{\ell},\underline{m}}. \tag{4}$$

Prove for $k = 2$ that

$$\|u - P_L u\|_{H^{\frac{1}{2}}_{\text{mix}}(D^k)} \leq \frac{1}{(L+2)^{s-\frac{1}{2}}} \|u\|_{H^s_{\text{iso}}(D^k)} \tag{5}$$

where

$$H^s_{\text{mix}}(D^k) := H^s(D) \otimes \ldots \otimes H^s(D) \tag{6}$$

2. **Sobolev Norm on a spheroid**
   Let $s \in \mathbb{R}$, D a spheroid and $v, w \in H^s(D)$. Then we have

$$\langle v, w \rangle_{H^s(D)} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1+\ell)^{2s} \hat{v}_{\ell,m} \hat{w}_{\ell,m}, \tag{7}$$

   where

$$v = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \hat{v}_{\ell,m} Y_{\ell,m}. \tag{8}$$

   Prove that for any $t \in \mathbb{R}$

$$\|v\|_{H^s(D)} = \sup_{w \in H^t(D)} \frac{\langle v, w \rangle_{H^{\frac{s+t}{2}}(D)}}{\|w\|_{H^t(D)}} \tag{9}$$

3. **Dimension of the sparse tensor product spectral discretization of a k-th moment equation**
   We call

$$\hat{S}_L := \operatorname{span}\left\{Y_{\underline{\ell},\underline{m}} : \ \underline{\ell} \in \delta_L \ ; \ |m_i| \leq \ell_i\right\} \tag{10}$$

   the sparse tensor product space, where

$$\delta_L := \left\{ \ell \in \mathbb{N}_0^k : \prod_{i=1}^{k} (1+\ell_i) \leq 1 + L \right\}. \tag{11}$$

   Prove for $k = 2$ **that**

$$\dim \hat{S}_L \leq C L^2 \log^{k-1}(L). \tag{12}$$

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Website: `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`