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Numerical Simulation (V4E2)
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Problem Sheet 7

1. A representation for the derivative of $\det(DT_t)$

Let $D \subset \mathbb{R}^d$ be a smooth domain. We define a smooth velocity

$$V: \begin{array}{rcl} \mathbb{R}_+ \times \mathbb{R}^d & \to & \mathbb{R}^d \\ (t,x) & \mapsto & V(t,x) \end{array} \tag{1}$$

with a compact support in $\mathbb{R}^d$. We can describe the displacement of a point $x \in \mathbb{R}^d$ by V through T_t, for which holds:

$$\left\{ \begin{array}{rcl} \frac{d}{dt}T_t(X) & = & V(t, T_t(X)), \\ T_0(X) & = & X. \end{array} \right. \tag{2}$$

By this, we can define perturbed domains of D via $D_t = T_t(D)$. Furthermore we define the transformation determinant $\gamma(t) := \det(DT_t)$, such that for a smooth function $Y : D_t \to \mathbb{R}$ we have

$$\int_{D_t} Y(x)\,dx = \int_D Y(T_t(x))\gamma(t,x)\,dx. \tag{3}$$

Prove for $d = 2$ that $\gamma'(0) = \operatorname{div} V(0)$.

2. **On the Poisson Equation with Neumann Boundary on a perturbed domain**
We look at the problem

$$\begin{cases} -\Delta u &= f \quad \text{in } D, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma = \partial D, \end{cases} \tag{4}$$

where f and g are functions defined on $\mathbb{R}^d$. There exists a unique solution in $H^1(D)/\mathbb{R}$. Let V and T_t be defined as in the first exercise. In the lecture we have defined the shape derivative of a function $y(D_t)$ as

$$y'(D,V) := \dot{y}(D,V) - \nabla y(D) \cdot V(0), \tag{5}$$

where the material derivative was defined by

$$\dot{y}(D,V) := \lim_{t \to 0} \frac{y(D_t) \circ T_t - y(D)}{t}. \tag{6}$$

The shape derivative of a function $z(\Gamma_t)$ on a surface is defined equivalently

$$z'(\Gamma,V) := \dot{z}(\Gamma,V) - \nabla_\Gamma z(\Gamma) \cdot V(0), \tag{7}$$

where the material derivative was defined by

$$\dot{z}(\Gamma,V) := \lim_{t \to 0} \frac{z(\Gamma_t) \circ T_t - z(\Gamma)}{t}. \tag{8}$$

Prove that $u' = u'(D,V) \in H^1(D)/\mathbb{R}$ **fullfills**

$$\begin{cases} -\Delta u' &= 0 & \text{in } D, \\ \frac{\partial u'}{\partial n} &= \operatorname{div}_\Gamma(v_n(0)\nabla_\Gamma u) + f v_n(0) + (\frac{\partial g}{\partial n} + g\kappa)v_n(0)) & \text{on } \Gamma = \partial D, \end{cases} \tag{9}$$

where $\kappa = \operatorname{div}_\Gamma(n)$ is the mean curvature of Γ and $v_n = V \cdot n$ is the velocity in the normal direction. By this equation u' is uniquely defined in $H^1(D)/\mathbb{R}$.

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