universitätbonn
Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn
hausdorff CENTER FOR MATHEMATICS

Numerical Simulation (V4E2)
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Prof. Dr. Alexey Chernov
Claudio Bierig

Problem Sheet 8

1. **A different a priori error estimate for the Monte Carlo Method**
   Let u be a function in $L^2(\Omega, V)$ for some Hilbertspace V and a probability space $(\Omega, \Sigma, \mathbb{P})$. We have proven the following a priori error estimate in the lecture: For any $M \in \mathbb{N}$
   $$\|\mathbb{E}[u] - E_M[u]\|_{L^2(\Omega,V)} \leq M^{-\frac{1}{2}} \|u\|_{L^2(\Omega,V)} \tag{1}$$
   holds, where we defined $E_M[u] = \frac{1}{M}\sum_{i=1}^{M} \hat{u}^i$ and $\hat{u}^i$ are random iid variables, copies of u.
   Prove that for any $0 < \varepsilon < 1$
   $$\mathbb{P}\left\{ \|\mathbb{E}[u] - E_M[u]\|_V \leq \frac{1}{\sqrt{\varepsilon M}} \|u\|_{L^2(\Omega,V)} \right\} \geq 1 - \varepsilon \tag{2}$$
   holds.
   Hint: Use
   $$\int_{\{\omega \in \Omega \,|\, \|Y(\omega)\|_V \geq \lambda\}} \|Y(\omega)\|_V^2 \, d\mathbb{P}(\omega) \geq \lambda^2 \mathbb{P}\{\|Y\|_V \geq \lambda\} \tag{3}$$
   and (1).

2. **On the Cost for computing k-th moments with the MLMC method**
Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{N}_0$ and $L \in \mathbb{N}$.
Prove that

$$\sum_{\ell=1}^{L} (L - \ell + 1)^\beta e^{\alpha \ell} \leq C(\alpha, \beta) \begin{cases} L^\beta & \alpha < 0 \\ L^{\beta+1} & \alpha = 0 \\ e^{\alpha L} & \alpha > 0 \end{cases} \tag{4}$$

holds.
The same sum appears in the computation of the cost of the MLMC method for the k-th moment. We gain the sum for $\beta = 2(k-1)$ and $\alpha = (2s - d) \log 2$.
Hint: Use Abel's summation formula: For any sequence $(a_n)_n \in \mathbb{R}^\infty$ and any function $\varphi \in C^1(\mathbb{R}, \mathbb{R})$ it holds, that

$$\sum_{1 \leq \ell \leq L} \varphi(\ell) a_\ell = A(L)\varphi(L) - \int_1^L A(t)\varphi'(t)\, dt, \tag{5}$$

where

$$A(x) := \sum_{0 < n \leq x} a_n. \tag{6}$$

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