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Topological Transfinite Interpolations in the Multidimensional Simplex and Hypercube

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Abstract: We first demonstrate how to construct a transfinite interpolation on a simplex of arbitrary dimension d. The inputs are $(d+1)$ functions describing the simplicial faces of decremented dimension $(d-1)$. We deduce the explicit expressions for tetrahedra and triangles by using the CGNS convention for the enumeration of topological entities. Second, our construction is applied to hypercubes as well. Rigorous coincidence with the usual Coons tensor product on hypercubes is theoretically shown. We prove the exact correlation between the blending functions using multidimensional barycentric coordinates and those using tensor product for the case of hypercubes. We conjecture that our approach holds true for any transfinite interpolation where the domain of definition is a convex polytope. We do not present any numerical results since this paper is only of theoretical nature.
Key Words: Simplex, Hypercube, Transfinite Interpolation.

1 Introduction

Transfinite interpolations on a reference domain $\mathcal{D}_{\rm ref}$ start by providing some facial functions on the boundary facets of $\mathcal{D}_{\rm ref}$. One seeks then a function defined in the interior of $\mathcal{D}_{\rm ref}$ which interpolates the initial facial functions. The main objective is to obtain simple short formulae without solving any equations. In the present paper, we want to address the problem of transfinite interpolation from the topological perspective. One of the most important relations in topology or solid modeling is the Euler identity [4] which gives the interdependence between the number of nodes, edges and higher dimensional entities in a solid. In fact, for the case of simplex and hypercube of dimension d, the Euler identity is

$$(-1)^d + 1 = \sum_{k=0}^{d-1} (-1)^k M_k \tag{1}$$

where M_k denotes the number of subsimplices $\boldsymbol{\sigma}_i^k$ or subcubes $\boldsymbol{\kappa}_i^k$ which are of dimension k. The usual transfinite interpolation as introduced by Coons does not contain any topological information whatsoever. In this document, we will investigate some formula resembling equation (1) which expresses a function within a simplex or a hypercube of dimension d (see Fig. 2) by using topological entities of lower dimensions.

Although topological structures are precious tools for theoretical analyses, they are useful in practical applications too. Topological organizations have already gained practical acceptance by many scientists for long time although that is often only implicitly stated. As a standard illustration, CGNS which stands for **C**FD **G**eneral **N**otation **S**ystem determines an enumeration of the topological entities inside a given cell. By using CGNS, you have standard data structures for enlisting the nodes, edges, faces of usual domains such as triangles, tetrahedra, pentahedra, hexahedra, to name only a few. Nowadays, the CGNS convention is already adopted by many practitioners treating mesh generations, CFD simulations, domain decompositions and solid modeling. Other important applications involve $\boldsymbol{\alpha}$-shapes which have considerably attracted attentions in computer graphics over the last decade and which are already treated in libraries like CGAL.

Related works are as follows. The author who initiated the idea of such transfinite interpolations was Steven Coons [3] in the mid 60's. His idea was improved by William Gordon by using some operator and Boolean sums [9, 10, 11]. Robin Forrest has intensively used transfinite interpolation in order to generate curved coordinate systems [8]. As for CAD preparation and triangular patch representations, known results are as follows. Brunnett and Randrianarivony have invested a lot to develop a method which is appropriate for surfaces in integral equations.

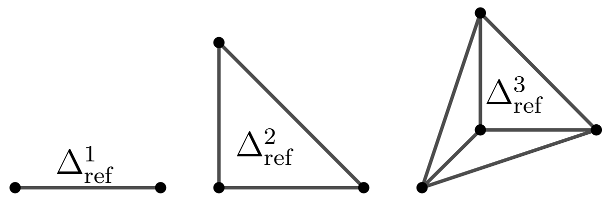


Figure 1: First simplices $\Delta^d_{\rm ref}$: unit interval, unit triangle, unit tetrahedron, unit pentachoron.

Their methods have already been successfully implemented to CAD and molecular surfaces [17, 18]. Harbrecht and Randrianarivony [12] have used those surface CAD models for applications in Wavelet BEM.

The present paper is structured as follows. The purpose of Section 2 is to recall the notion of transfinite interpolation on the unit square in order to motivate the next discussion. Additionally, we will find there a specification of the problem formulation and introduction of important definitions. The main results of this paper can be found in Section 3 and Section 4. In fact, Section 3 treats the case of higher dimensional simplices. In fact, the formula for transfinite interpolation in a tetrahedron takes already a dozen of lines. As a consequence, the author introduces in that section a succinct and compact way of representing such formula for multi-dimensional simplices. In section 4, we concentrate on the case of the multidimensional hypercube. In particular, we investigate the relation between our proposed formula and the usual transfinite interpolation which uses Boolean sums of operators.

2 Generality and Problem Setting

2.1 Motivation from the Usual 2D Coons Map

In order to simplify the description about multidimentional transfinite interpolations, we will recall the usual Coons patch [3] in this section. Let us consider four parametric curves

$$\boldsymbol{\alpha},\, \boldsymbol{\beta},\, \boldsymbol{\gamma},\, \boldsymbol{\delta} \,:[0,1] \longrightarrow \mathbb{R}^2 \tag{2}$$

which are supposed to fulfil the next *compatibility condition* which are illustrated in Fig. 3(a):

$$\boldsymbol{\alpha}(0)=\boldsymbol{\delta}(0)\,, \qquad \boldsymbol{\alpha}(1)=\boldsymbol{\beta}(0)\,, \qquad \boldsymbol{\gamma}(0)=\boldsymbol{\delta}(1)\,, \qquad \boldsymbol{\gamma}(1)=\boldsymbol{\beta}(1). \tag{3}$$

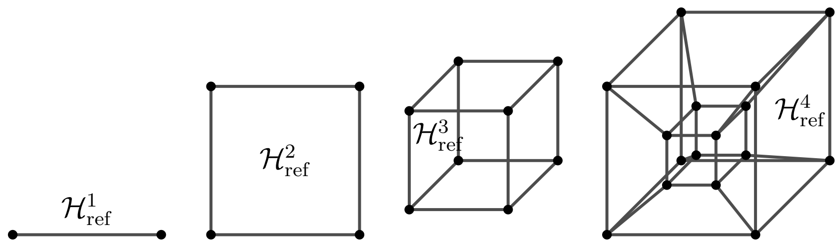
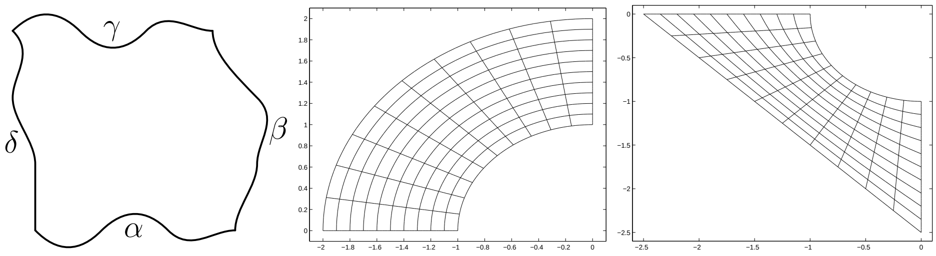


Figure 2: First hypercubes $\mathcal{H}^d_{\rm ref}$: unit interval, unit square, unit cube, unit tesseract.



(a) (b) (c)

Figure 3: (a)Boundary of a Coons patch, (b),(c) Examples of Coons patches

Type	$f_0(t)$	$f_1(t)$
Linear	$1-t$	t
Cubic	$B_0^3(t)+B_1^3(t)$	$B_2^3(t)+B_3^3(t)$
Trigonometric	$\cos^2(0.5\pi t)$	$\sin^2(0.5\pi t)$

Table 1: Blending functions.

We are interested in generating a parametric surface $\mathbf{x}(u,v)$ defined on the unit square $[0,1]^2$ such that the boundary of the image of $\mathbf{x}$ coincides with the given four curves:

$$\begin{array}{lll} \mathbf{x}(u,0)=\boldsymbol{\alpha}(u) & \mathbf{x}(u,1)=\boldsymbol{\gamma}(u) & \forall\, u\in[0,1] \\ \mathbf{x}(0,v)=\boldsymbol{\delta}(v) & \mathbf{x}(1,v)=\boldsymbol{\beta}(v) & \forall\, v\in[0,1]\ . \end{array} \tag{4}$$

Let f_0 and f_1 denote two arbitrary smooth functions satisfying

$$f_i(j)=\delta_{ij} \quad i,j=0,1 \quad \text{and} \qquad f_0(t)+f_1(t)=1 \quad \forall\, t\in[0,1]. \tag{5}$$

The functions f_0, f_1 which are better known as *blending functions* can be chosen in several ways [8]. Among others, three methods are shown in Table 1 for choosing them. A construction of a solution to (4) which was due to S. Coons can be expressed in matrix form as:

$$\begin{aligned} \mathbf{x}(u,v) \;=\; & \begin{bmatrix} f_0(u) & f_1(u) \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}(v) \\ \boldsymbol{\beta}(v) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\alpha}(u) & \boldsymbol{\gamma}(u) \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \end{bmatrix} - \\ & \begin{bmatrix} f_0(u) & f_1(u) \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}(0) & \boldsymbol{\gamma}(0) \\ \boldsymbol{\alpha}(1) & \boldsymbol{\gamma}(1) \end{bmatrix} \begin{bmatrix} f_0(v) \\ f_1(v) \end{bmatrix}. \end{aligned} \tag{6}$$

As an illustration in Fig. 3(b) and Fig. 3(c), we can see the image of an uniform grid on the unit square by a Coons map $\mathbf{x}$. In the remainder of this paper, we will generate transfinite interpolations on two more general domains: the simplex $\Delta^d_{\rm ref}$ and the hypercube $\mathcal{H}^d_{\rm ref}$. Similarly to the 2D case, we need three ingredients to ensure the generation of transfinite interpolation in the multidimensional case:

- Functions on the facets as in (2),
- Blending functions generalizing(5),
- Compatibility conditions generalizing (3).

2.2 Nomenclature Related to $\Delta^d_{\rm ref}$ and $\mathcal{H}^d_{\rm ref}$

We consider an Euclidean space which is suppose to be $\mathbb{R}^n$ accompanied with the usual Euclidean distance. Consider a convex domain $\mathcal{D}_{\rm ref}$ in $\mathbb{R}^n$. It is supposed to be the convex hull of the vertices $N_i \in \mathbb{R}^n$ for $i \in \mathcal{J}$ where $\mathcal{J}$ is some finite index set. *Barycentric coordinates* will be a set of functions $\Lambda(\mathbf{u}) = (\lambda_i(\mathbf{u}))_{i\in\mathcal{J}}$ such that

$$\mathbf{u} = \sum_{i\in\mathcal{J}} \lambda_i(\mathbf{u}) N_i \qquad \text{and} \qquad \sum_{i\in\mathcal{J}} \lambda_i(\mathbf{u}) = 1 \qquad \forall\, \mathbf{u} \in \mathcal{D}_{\mathrm{ref}}, \tag{7}$$

$$\forall\, (\mu_i)_{i\in\mathcal{J}} \quad \text{with} \quad \sum_{i\in\mathcal{J}} \mu_i(\mathbf{u}) = 1, \quad \exists!\, \mathbf{u} \in \mathcal{D}_{\mathrm{ref}} \text{ where } \lambda_i(\mathbf{u}) = \mu_i \quad \forall\, i \in \mathcal{J}. \tag{8}$$

Because of those two properties, for any function G defined on $\mathcal{D}_{\mathrm{ref}}$, we will write interchangeably $G(\Lambda)$ and $G(\mathbf{u})$ which is in fact the composition of G and Λ. For each $p \in \mathcal{J}$, the barycentric coordinates which take zero value except at the p-th entry which contains unity will be denoted by

$$\Lambda^p_n := (\lambda^p_k)_{k\in\mathcal{J}} \qquad \text{where} \qquad \lambda^p_k = \delta_{p,k} \qquad \text{for} \qquad p, k \in \mathcal{J}. \tag{9}$$

In the sequel, we will treat only the case where $\mathcal{D}_{\mathrm{ref}}$ is a the unit simplex Δ^d_{ref} or the unit hypercube $\mathcal{H}^d_{\mathrm{ref}}$ which we want to introduce now.
In order to introduce the multidimensional simplex Δ^d_{ref}, let us use $N_0 := (0, 0, ..., 0) \in \mathbb{R}^d$, $N_1 := (1, 0, ..., 0) \in \mathbb{R}^d$, $N_2 := (0, 1, ..., 0) \in \mathbb{R}^d, \cdots$, $N_d := (0, 0, ..., 1) \in \mathbb{R}^d$. The set of vertices is then

$$\mathcal{A} := \{N_i \in \mathbb{R}^d, \quad i = 0, ..., d\}. \tag{10}$$

The *multidimensional simplex* Δ^d_{ref} is the hull of them which is

$$\Delta^d_{\mathrm{ref}} := \{\mathbf{u} = (u_1, ..., u_d) \in \mathbb{R}^d : \quad u_i \geq 0, \quad u_1 + \cdots + u_d \leq 1\}. \tag{11}$$

A *subsimplex* $\boldsymbol{\sigma}$ of Δ^d_{ref} is the convex hull of points which belong to a subset of $\mathcal{A}$. The *dimension* of the subsimplex $\boldsymbol{\sigma}$ is the decremented number of vertices $N_i \in \mathcal{A}$ which are in $\boldsymbol{\sigma}$. The simplices of dimension k of Δ^d_{ref} will be denoted by $\boldsymbol{\sigma}^k_i$ in which $i = 1, 2, ..., \eta_k$ where $\eta_k := \binom{d+1}{k}$ is the number of k-simplices. The subsimplices of dimension $d-1$ which are $\boldsymbol{\sigma}^{d-1}_i$ are the *facets* of Δ^d_{ref} and those of dimension zero are the initial vertices $\boldsymbol{\sigma}^0_i = N_i$. Simplices of dimension one are called *edges*.
Let us consider now the case of hypercubes. To facilitate the presentation, we will use the following set of integers

$$[\![a, b]\!] := \{z \in \mathbb{Z} \quad : \qquad a \leq z \leq b\}. \tag{12}$$

We introduce the set of vertices to be

$$\mathcal{C} := \left\{\mathbf{u} = (u_1, ..., u_d) \in \mathbb{R}^d \quad \text{where} \quad u_i \in \{0, 1\} \quad \text{for} \quad i = 1, ..., d\right\}. \tag{13}$$

The *multidimensional hypercube* $\mathcal{H}^d_{\mathrm{ref}}$ is the convex hull of $\mathcal{C}$ which is

$$\mathcal{H}^d_{\mathrm{ref}} := [0, 1]^d. \tag{14}$$

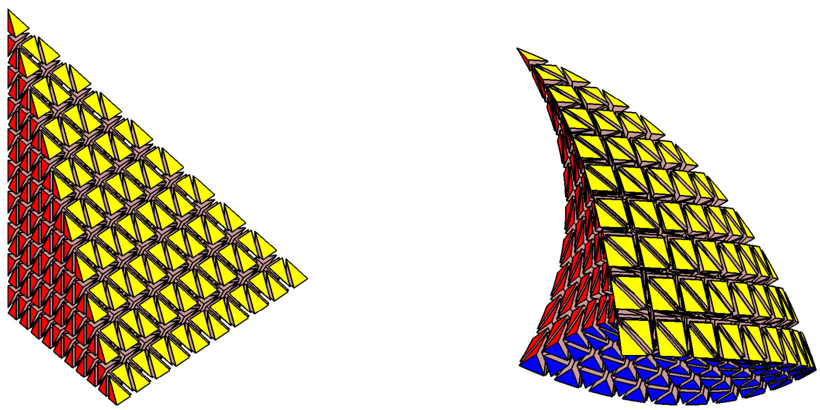
For any subset $\xi \subset \{1, ..., d\}$ and a binary function $\phi : \xi \to \{0, 1\}$, we define

$$\mathcal{R}(\xi, \phi) := \{(u_1, ..., u_d) \in \mathbb{R}^d \quad \text{where} \quad u_j = \phi(j) \quad \text{for} \quad j \in \xi\}. \tag{15}$$

A *subcube* $\boldsymbol{\kappa}$ is any intersection of $\mathcal{H}^d_{\mathrm{ref}}$ with some $\mathcal{R}(\xi, \phi)$. The *dimension* of $\boldsymbol{\kappa}$ is the number of non-fixed variables in $\mathcal{R}(\xi, \phi)$: that is $\dim(\boldsymbol{\kappa}) := d - \mathrm{Card}(\xi)$. Consider the following set

$$\mathcal{J}^k := \{\boldsymbol{\alpha} = (\xi, \phi) : \quad \xi \subset [\![1, d]\!], \quad \mathrm{Card}(\xi) = d - k, \quad \phi : \xi \to \{0, 1\}\} \tag{16}$$

which is a finite set where the number of elements is $\mathrm{Card}(\mathcal{J}^k) = \binom{d}{d-k} 2^{d-k}$. The k-dimensional subcubes of $\mathcal{H}_{\mathrm{ref}}$ will be enlisted as $(\boldsymbol{\kappa}^k_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathcal{J}^k}$. The subcubes of dimension $d-1$ which are $\boldsymbol{\kappa}^{d-1}_{\boldsymbol{\alpha}}$ are the *facets* of $\mathcal{H}^d_{\mathrm{ref}}$.



(a) (b)

Figure 4: (a)Tetrahedralization of the reference tetrahedron Δ^3_{ref} (b)Image by a tetrahedral transfinite interpolation of the left-hand mesh.

In the sequel, we denote the convex hull of a set S of points $X_i \in \mathbb{R}^n$ where $i = 0, ..., M$ by

$$\begin{aligned} \text{Conv}(S) &= \text{Conv}[X_0, ..., X_M] & (17)\\ &:= \left\{ \sum_{i=0}^{M} \mu_i X_i \quad : \quad \sum_{i=0}^{M} \mu_i = 1, \quad \mu_i \in [0,1] \right\}. & (18) \end{aligned}$$

The purpose of this document is to search for a transfinite interpolation. That is, if we are given some functions on the facets of Δ^d_{ref} or $\mathcal{H}^d_{\text{ref}}$, then we search for a function defined on the reference elements which interpolates those initial facial functions.
To generalize the case of 2D Coons map, let us introduce the notion of barycentric blending functions b_i, $i \in \mathcal{J}$ defined for all $\Lambda = (\lambda_i)_{i\in\mathcal{J}}$ in the reference domain. In fact, we define them to be a set of functions verifying the following three properties:

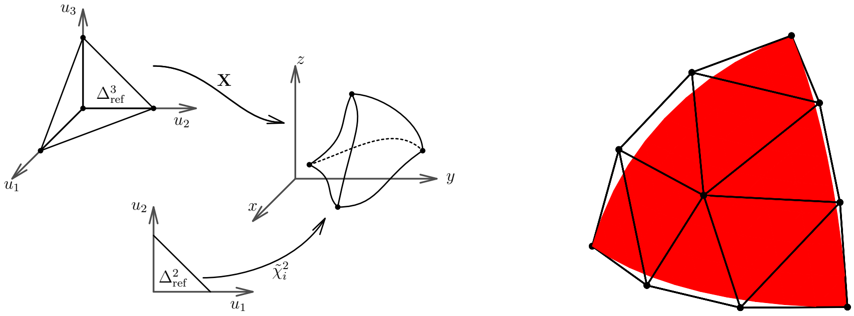
$$\left\{ \begin{array}{lll} (P1): & \text{If} \quad \lambda_p = 0 \quad \text{in} \quad \Lambda = (\lambda_i)_{i\in\mathcal{J}} \quad \text{then} \quad b_p(\Lambda) = 0, \\ (P2): & b_p(\Lambda^p_n) = 1, \quad \text{for all} \quad p \in \mathcal{J}, \\ (P3): & \sum_{i\in\mathcal{J}} b_i(\Lambda) = 1. \end{array} \right. \tag{19}$$

As in the case of 2D transfinite interpolation, the simplest way of choosing the blending functions (see Table 1) is to use the linear ones which are in our case:

$$b_i(\Lambda) := \lambda_i, \qquad \forall\, i \in \mathcal{J}. \tag{20}$$

3 Interpolation for multidimensional simplices

In this section, we concentrate on transfinite interpolation within a multidimensional simplex Δ^d_{ref}. First, we will introduce the induced mapping $\tilde{\chi}^k_j$ defined on each subsimplex $\boldsymbol{\sigma}^k_j$. Second, we will use a certain type of projection with which help the transfinite interpolation is expressed. After proving the interpolation of boundary faces, we show eventually two particular cases: the triangular and the tetrahedral transfinite interpolations.



(a) (b)

Figure 5: (a)Tetrahedral transfinite interpolation (b)A simplex Bézier and its control net embedded in $\mathbb{R}^3$

3.1 Relations between the subsimplices

The following exact expression of barycentric coordinates in simplex will not be used explicitly but we introduce them anyway for sake of completeness. We recall the barycentric coordinates with respect to $W = (\mathbf{w}_0, \cdots, \mathbf{w}_d)$ where $\mathbf{w}_i \in \mathbb{R}^d$ by using

$$D(W) := \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ \mathbf{w}_0 & \mathbf{w}_1 & \dots & \mathbf{w}_d \end{bmatrix}. \tag{21}$$

Additionally, we define for any $\mathbf{u} \in \mathbb{R}^d$

$$D_i(W|\mathbf{u}) := \det \begin{bmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \mathbf{w}_0 & \dots & \mathbf{w}_{i-1} & \mathbf{u} & \mathbf{w}_{i+1} & \dots & \mathbf{w}_d \end{bmatrix}. \tag{22}$$

The barycentric coordinates of $\mathbf{u}$ with respect to W are defined by

$$\lambda_i(\mathbf{u}) := D_i(W|\mathbf{u})/D(W). \tag{23}$$

As first step, we assign an enumeration to the vertices of each k-simplex $\boldsymbol{\sigma}^k_i$ of Δ^d_{ref}. Suppose that the nodes of $\boldsymbol{\sigma}^k_i$ are N_p where $p \in \mathcal{A}^k_i$ a subset of cardinality $k+1$ of $\mathcal{A}$ from relation (10). The enumeration is arbitrary but fixed meaning that we fix a bijection

$$\psi^k_i : [\![0, k]\!] \longrightarrow \mathcal{A}^k_i \tag{24}$$

which is always possible because $\mathcal{A}^k_i$ is a discrete set of cardinality $k+1$. Thus, the vertices of $\boldsymbol{\sigma}^k_i$ are

$$N_{\psi^k_i(0)}, N_{\psi^k_i(1)}, \cdots, N_{\psi^k_i(k)}. \tag{25}$$

By using that, we can deduce a function χ^k_j defined for all $\Lambda = (\lambda_0, ..., \lambda_k) \in \Delta^k_{\mathrm{ref}}$ such that $\chi^k_j(\Delta^k_{\mathrm{ref}}) = \boldsymbol{\sigma}^k_j$ as follows.

$$\chi^k_j(\Lambda^q) = \chi^k_j(0, ..., 0, 1, 0, ..., 0) := N_{\psi^k_j(q)}, \tag{26}$$

$$\chi^k_j(\Lambda) = \chi^k_j(\lambda_0, ..., \lambda_k) := \sum_{q=0}^{k} \lambda_q N_{\psi^k_j(q)}. \tag{27}$$

The general transfinite interpolation problem consists in giving $(d+1)$ functions defined on $\Delta^{d-1}_{\mathrm{ref}}$:

$$\tilde{\chi}^{d-1}_i : \Delta^{d-1}_{\mathrm{ref}} \to \mathbb{R}^d \quad \text{where} \quad \tilde{\boldsymbol{\sigma}}^{d-1}_i := \tilde{\chi}^{d-1}_i(\Delta^{d-1}_{\mathrm{ref}}). \tag{28}$$

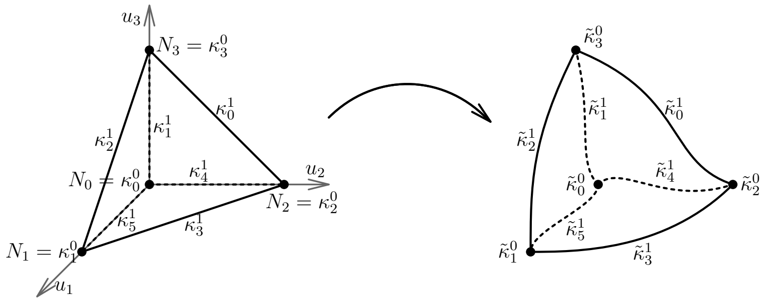


Figure 6: Only subsimplices $\boldsymbol{\sigma}_i^k$ of dimension $k \leq 1$ are labelled

From this we can induces some mappings $\tilde{\chi}_i^{k-1} : \Delta_{\text{ref}}^{d-1} \to \widetilde{\boldsymbol{\sigma}}_i^{k-1}$ for all subsimplices of lower dimension as follows. Suppose $\boldsymbol{\sigma}_i^{k-1}$ is a subsimplex of $\boldsymbol{\sigma}_j^k$. Hence, there is only one node of $\boldsymbol{\sigma}_j^k = \text{Conv}[N_{\psi_j^k(0)}, ..., N_{\psi_j^k(k)}]$ which is not in $\boldsymbol{\sigma}_i^{k-1} = \text{Conv}[N_{\psi_i^{k-1}(0)}, ..., N_{\psi_i^{k-1}(k-1)}]$. Let $N_{\psi_j^k(q)}$ be such a node. Thus, one can define a mapping $M_{i,j}^k : [\![1, p]\!] \setminus \{q\} \to [\![1, k-1]\!]$ such that

$$\psi_j^k(p) = \psi_i^{k-1}[M_{i,j}^k(p)] \quad \text{for each} \quad p = 0, ..., k \quad \text{such that} \quad p \neq q. \tag{29}$$

The induced function for $\boldsymbol{\sigma}_i^{k-1}$ is defined as

$$\tilde{\chi}_i^{k-1}(\mu_0, ..., \mu_{k-1}) := \tilde{\chi}_j^k(\lambda_0, ..., \lambda_k), \tag{30}$$

where $\lambda(s) := \mu_{M_{i,j}^k(s)}$ if $s \neq q$ and $\lambda(q) = 0$. Throughout this paper, we use tilde for expressions relative to the image entities such as the ones in Fig. 6. As we have done in relation (3) for the 2D case, we need some compatibility conditions about the given functions $\tilde{\chi}_i^{d-1}$. Here, that means that if $\boldsymbol{\sigma}_i^{k-1}$ is a subsimplex of two different simplices $\boldsymbol{\sigma}_{j_1}^k$ and $\boldsymbol{\sigma}_{j_2}^k$, then both induced functions coincide. For instance, in the case of tetrahedra, the curved edge at the interface between two curved triangles should have the same parametrization from both sides (see Fig. 4(a)).

3.2 Projections on subsimplices

Consider now an arbitrary subsimplex $\boldsymbol{\sigma} := \boldsymbol{\sigma}_j^k$ which contains the node N_i and let us use the mapping $\tilde{\chi}_j^k : \Delta_{\text{ref}}^k \to \widetilde{\boldsymbol{\sigma}}_j^k$. Thus, there is an index p such that $\psi_j^k(p) = N_i$. For any $\Lambda = (\lambda_0, ..., \lambda_d) \in \Delta_{\text{ref}}^d$, we introduce

$$\mathcal{F}_{\boldsymbol{\sigma}, N_i}(\Lambda) := \tilde{\chi}_j^k(\lambda_{\psi_j^k(0)}, ..., \lambda_{\psi_j^k(p-1)}, 1 - \sum_{\substack{s=0 \\ s \neq p}}^{k} \lambda_{\psi_j^k(s)}, \lambda_{\psi_j^k(p+1)}, ..., \lambda_{\psi_j^k(k)}). \tag{31}$$

Lemma 3.1 *For every $d \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{d-1} (-1)^k \binom{d}{k} = (-1)^{d+1}. \tag{32}$$

PROOF.
This is proved by induction with respect to d. For $d = 1$, the sum is $1 = (-1)^{1+1}$. Suppose that it is the case for d. For $d + 1$ we have:

$$\begin{aligned}\sum_{k=0}^{d}(-1)^k\binom{d+1}{k} &= \sum_{k=0}^{d}(-1)^k\left[\binom{d}{k}+\binom{d}{k-1}\right]\\ &= (-1)^d+\sum_{k=0}^{d-1}(-1)^k\binom{d}{k}+\sum_{k=1}^{d}(-1)^k\binom{d}{k-1}=(-1)^{d+2}.\end{aligned}$$

Q.E.D.

Lemma 3.2 *Let N_q be a fixed node of the d-dimensional simplex Δ^d_{ref} and consider the $(d-1)$-simplex $\boldsymbol{\sigma}_{\mathrm{opp}} \subset \Delta^d_{\mathrm{ref}}$ which does not contain N_q. For any $\Lambda = (\lambda_0, ..., \lambda_d)$ where $\lambda_q = 0$, the following relation holds independently of $i \in [\![0, d]\!]$ where $i \neq q$:*

$$\mathcal{F}_{\boldsymbol{\sigma}_{\mathrm{opp}},N_i}(\Lambda) = \tilde{\chi}^{d-1}_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)}) \tag{33}$$

for some $e = 0, ..., d$.
Additionally, for any subsimplex $\boldsymbol{\sigma}^k_j$ containing both N_q and N_i, there is a subsimplex $\boldsymbol{\sigma}^{k-1}_p \subset \boldsymbol{\sigma}^k_j$ such that

$$(-1)^{k-1}\mathcal{F}_{\boldsymbol{\sigma}^{k-1}_p,N_i}(\Lambda) + (-1)^k\mathcal{F}_{\boldsymbol{\sigma}^k_j,N_i}(\Lambda) = 0. \tag{34}$$

PROOF.
Since $\boldsymbol{\sigma}_{\mathrm{opp}}$ is of dimension $d-1$, there exist some e such that $\boldsymbol{\sigma}_{\mathrm{opp}} = \boldsymbol{\sigma}^{d-1}_e$. We want to show that $\mathcal{F}_{\boldsymbol{\sigma}_{\mathrm{opp}},N_i}$ is independent of N_i. Because of partition of unity and $\lambda_q = 0$, we have for the p in (31):

$$1 - \sum_{\substack{s\in[\![0,p-1]\!]\\ s\neq p}} \lambda_{\psi^{d-1}_e(s)} = \lambda_{\psi^{d-1}_e(p)} + \lambda_q = \lambda_{\psi^{d-1}_e(p)}. \tag{35}$$

As a consequence, we obtain for $\boldsymbol{\sigma}_{\mathrm{opp}}$:

$$\begin{aligned}\mathcal{F}_{\boldsymbol{\sigma}_{\mathrm{opp}},N_i}(\Lambda) &= \tilde{\chi}^{d-1}_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(p-1)}, \lambda_{\psi^{d-1}_e(p)}, \lambda_{\psi^{d-1}_e(p+1)}, ..., \lambda_{\psi^{d-1}_e(d-1)})\\ &= \tilde{\chi}^{d-1}_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)})\end{aligned}$$

which proves (33).
Consider the subsimplex $\boldsymbol{\sigma}^{k-1}_p$ of dimension $k-1$ which is obtained by excluding N_q from $\boldsymbol{\sigma}^k_j$. Since N_q and N_i belong to $\boldsymbol{\sigma}^k_j$, there are some w, p such that $N_q = \psi^k_j(w)$ and $N_i = \psi^k_j(p)$.

$$\begin{aligned}\mathcal{F}^k_{\boldsymbol{\sigma}_j,N_i}(\Lambda) &= \tilde{\chi}^k_j(\lambda_{\psi^k_j(0)}, ..., \lambda_{\psi^k_j(w-1)}, \lambda_q, \lambda_{\psi^k_j(w+1)}, ..., \lambda_{\psi^k_j(r-1)}, T, \lambda_{\psi^k_j(r+1)}, ..., \lambda_{\psi^k_j(k)})\\ &= \tilde{\chi}^k_j(\lambda_{\psi^k_j(0)}, ..., \lambda_{\psi^k_j(w-1)}, 0, \lambda_{\psi^k_j(w+1)}, ..., \lambda_{\psi^k_j(p-1)}, T, \lambda_{\psi^k_j(p+1)}, ..., \lambda_{\psi^k_j(k)}),\end{aligned}$$

in which $T := 1 - \sum_{s\neq p} \lambda_{\psi^k_j(s)} = 1 - \sum_{s\neq p,w} \lambda_{\psi^k_j(s)}$. Let us denote by $M := M^k_{p,j}$ the mapping that we met in (29). Hence,

$$\begin{aligned}\mathcal{F}^k_{\boldsymbol{\sigma}_j,N_i}(\Lambda) &= \tilde{\chi}^k_j(\lambda_{\psi^{k-1}_p[M(0)]}, ..., 0, ..., \lambda_{\psi^{k-1}_p[M(r-1)]}, T, \lambda_{\psi^{k-1}_p[M(r+1)]}, ..., \lambda_{\psi^{k-1}_p[M(k)]})\\ &= \tilde{\chi}^{k-1}_p(\lambda_{\psi^{k-1}_p(0)}, ..., \lambda_{\psi^{k-1}_p(r-1)}, T, \lambda_{\psi^{k-1}_p(r+1)}, ..., \lambda_{\psi^{k-1}_p(k-1)}).\end{aligned}$$

That is, we have $\mathcal{F}^{k-1}_{\boldsymbol{\sigma}_p,N_i}(\Lambda) = \mathcal{F}^k_{\boldsymbol{\sigma}_j,N_i}(\Lambda)$ which implies (34).

Q.E.D.

Now, we are ready to state the main result of this section.

Theorem 3.3 *Consider some barycentric blending functions $b_i(\Lambda)$ as in (19). For the d-dimensional simplex Δ^d_{ref}, the function*

$$\mathcal{T}(\Lambda) := (-1)^{d+1}\sum_{i=0}^{d} b_i(\Lambda)\sum_{\boldsymbol{\sigma}\in S_i}(-1)^{\dim(\boldsymbol{\sigma})}\mathcal{F}_{\boldsymbol{\sigma},N_i}(\Lambda) \tag{36}$$

where S_i is the set of all subsimplices $\boldsymbol{\sigma}^k_j$ containing the node N_i is a transfinite interpolation. That is, $\mathcal{T}(\Lambda)$ verifies for each $q = 0, ..., d$:

$$\begin{cases} \mathcal{T}(\Lambda^q_d) = N_q, \\ \mathcal{T}(\lambda_0, ..., \lambda_{q-1}, 0, \lambda_{q+1}, ..., \lambda_d) = \tilde{\chi}^{d-1}_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)}) \end{cases} \tag{37}$$

for some $e = 0, ..., d$ where $\boldsymbol{\sigma}^{d-1}_e$ is the $(d-1)$-subsimplex not containing N_q.

PROOF.
Let us prove the first equality of (37) by considering a node N_q for a fixed $q = 0, ..., d$. Due to property (P2) of the barycentric blending function in (19), the term $b_i(\Lambda^q_d)$ vanishes for $i \neq q$ and relation (36) becomes

$$\mathcal{T}(\Lambda^q_d) = (-1)^{d+1}\sum_{\boldsymbol{\sigma}\in S_q}(-1)^{\dim(\boldsymbol{\sigma})}\mathcal{F}_{\boldsymbol{\sigma},N_q}(\Lambda^q_d). \tag{38}$$

First, we will show that in the case $\Lambda = \Lambda^q$, the expression $\mathcal{F}_{\boldsymbol{\sigma},N_j}(\Lambda^q)$ is a constant $K = \tilde{N}_q$ independent of such $\boldsymbol{\sigma}$. Then, we will apply Lemma 3.1 to obtain the first relation of (37). Consider any $\boldsymbol{\sigma} = \boldsymbol{\sigma}^k_j$ which is incident upon N_q. Since the elements of Λ^q are zero except at the q-th entry, relation (31) yields

$$\mathcal{F}_{\boldsymbol{\sigma},N_q} = \tilde{\chi}^k_j(0, ..., 0, 1, 0, ..., 0) = \tilde{N}_q =: K. \tag{39}$$

Denote by S^k_q the set of k-subsimplices $\boldsymbol{\sigma}^k_j$ containing the node N_q so that the set of all subsimplices incident upon N_p can be organized as $S_q = \bigcup_{k=0}^{d-1} S^k_q$. Since every element $\boldsymbol{\sigma}^k_j$ of S^k_q contains N_q, it must be of the form

$$\boldsymbol{\sigma}^k_i = \text{Conv}\left[N_{\psi^k_i(1)}, ..., N_{\psi^k_i(s-1)}, N_p, N_{\psi^k_i(s+1)}, ..., N_{\psi^k_i(k+1)}\right] \tag{40}$$

where $N_p = N_{\psi^k_j(s)}$ for some $s = 1, ..., k+1$. Since the set $\{N_{\psi^k_i(1)}, ..., N_{\psi^k_i(s-1)}, N_{\psi^k_i(s+1)}, ..., N_{\psi^k_i(k+1)}\}$ is any k-subset of $\mathcal{A}$ from relation (10), there are $\binom{\text{card}(\mathcal{A})}{k} = \binom{d+1}{k}$ ways of choosing it. Thus, we have

$$\text{Card}(S_q) = \sum_{k=0}^{d-1}\text{Card}(S^k_q) = \sum_{k=0}^{d-1}\binom{d+1}{k}. \tag{41}$$

As a consequence to Lemma 3.1, we obtain

$$\mathcal{T}(\Lambda^q) = (-1)^{d+1}K\sum_{k=0}^{d-1}(-1)^k\binom{d+1}{k} = K = \tilde{N}_p \tag{42}$$

which proves the first equality in relation (37).
Let us now prove the second relation of (37) by fixing any $\Lambda = (\lambda_1, ..., \lambda_{q-1}, 0, \lambda_{q+1}, ..., \lambda_n)$. Thus, we obtain from relation (33) of Lemma 3.2 and relation (36) that

$$\begin{aligned} \mathcal{T}(\Lambda) &= (-1)^{d+1}\sum_{i\neq q} b_i(\Lambda)(-1)^{d+1}\tilde{\chi}^k_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)}) \\ &= (-1)^{d+1}(-1)^{d+1}\left(\sum_{i\neq q} b_i(\Lambda)\right)\tilde{\chi}^k_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)}) \\ &= \tilde{\chi}^k_e(\lambda_{\psi^{d-1}_e(0)}, ..., \lambda_{\psi^{d-1}_e(d-1)}) \end{aligned}$$

Q.E.D.

3.3 Particular cases using CGNS enumeration

In this section, we consider the practical use of the previous formula. Although this section seems to be direct application of the above theory, it is recommended that readers consider it because it gives an explanation of the abstractions and help understanding our method more deeply. As an illustration of formula (36), let us consider the case of triangles and tetrahedra. It is possible to enumerate topological entities arbitrarily but let us adopt the CGNS ordering [2] which is used by many practitioners except that we shift the numbers so that we start from 0 (instead of 1). We will consider only linear blending functions where $b_i(\Lambda) = \lambda_i$. As in the previous theory, all the 0-simplices are $\boldsymbol{\sigma}_i^0 = N_i$.

3.3.1 Tetrahedral transfinite interpolation

In the case of tetrahedra, the CGNS enumeration of the topological entities which are illustrated in Fig. 7(b) is as follows. The four triangular faces have the nodes

$$\begin{array}{lll} \boldsymbol{\sigma}_0^2 & \longrightarrow & [N_0, N_2, N_1], \\ \boldsymbol{\sigma}_2^2 & \longrightarrow & [N_1, N_2, N_3], \end{array} \qquad \begin{array}{lll} \boldsymbol{\sigma}_1^2 & \longrightarrow & [N_0, N_1, N_3], \\ \boldsymbol{\sigma}_3^2 & \longrightarrow & [N_2, N_0, N_3]. \end{array} \tag{43}$$

Additionally, the six edges are enumerated as follows

$$\begin{array}{lll} \boldsymbol{\sigma}_0^1 & \longrightarrow & [N_0, N_1], \\ \boldsymbol{\sigma}_2^1 & \longrightarrow & [N_2, N_0], \\ \boldsymbol{\sigma}_4^1 & \longrightarrow & [N_1, N_3], \end{array} \qquad \begin{array}{lll} \boldsymbol{\sigma}_1^1 & \longrightarrow & [N_1, N_2], \\ \boldsymbol{\sigma}_3^1 & \longrightarrow & [N_0, N_3], \\ \boldsymbol{\sigma}_5^1 & \longrightarrow & [N_2, N_3]. \end{array} \tag{44}$$

We are initially given four surfaces S_i for $i = 0, 1, 2, 3$ which are defined on the unit triangle Δ^2_{ref}. They are supposed to fulfil the next six compatibility conditions. For all (μ_0, μ_1) such that $\mu_0 + \mu_1 = 1$, we have:

$$\begin{array}{lll} S_0(\mu_0, 0, \mu_1) & = & S_1(\mu_0, \mu_1, 0), \\ S_0(0, \mu_1, \mu_0) & = & S_2(\mu_0, \mu_1, 0), \\ S_0(\mu_1, \mu_0, 0) & = & S_3(\mu_0, \mu_1, 0), \end{array} \qquad \begin{array}{lll} S_1(\mu_0, 0, \mu_1) & = & S_3(0, \mu_0, \mu_1), \\ S_1(0, \mu_0, \mu_1) & = & S_2(\mu_0, 0, \mu_1), \\ S_2(0, \mu_0, \mu_1) & = & S_3(\mu_0, 0, \mu_1). \end{array} \tag{45}$$

As a consequence to the enumeration in (43), we obtain

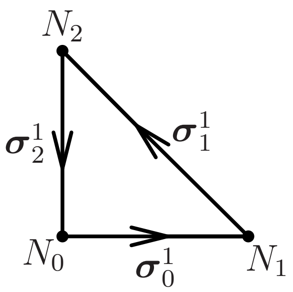
$$\begin{array}{lll} \psi_0^2(0) = 0 & \psi_0^2(1) = 2 & \psi_0^2(2) = 1 \\ \psi_1^2(0) = 0 & \psi_1^2(1) = 1 & \psi_1^2(2) = 3 \\ \psi_2^2(0) = 1 & \psi_2^2(1) = 2 & \psi_2^2(2) = 3 \\ \psi_3^2(0) = 2 & \psi_3^2(1) = 0 & \psi_3^2(2) = 3. \end{array} \tag{46}$$

Similarly, from (3.3.1) we obtain

$$\begin{array}{llll} \psi_0^1(0) = 0 & \psi_0^1(1) = 1, & \psi_1^1(0) = 1 & \psi_1^1(1) = 2, \\ \psi_2^1(0) = 2 & \psi_2^1(1) = 0, & \psi_3^1(0) = 0 & \psi_3^1(1) = 3, \\ \psi_4^1(0) = 1 & \psi_4^1(1) = 3, & \psi_5^1(0) = 2 & \psi_5^1(1) = 3. \end{array} \tag{47}$$

As a consequence, we have the following induced mappings

$$\tilde{\chi}_0^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_0^2(\mu_0, 0, \mu_1) = \tilde{\chi}_1^2(\mu_0, \mu_1, 0), \tag{48}$$
$$\tilde{\chi}_1^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_0^2(0, \mu_1, \mu_0) = \tilde{\chi}_2^2(\mu_0, \mu_1, 0), \tag{49}$$
$$\tilde{\chi}_2^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_0^2(\mu_1, \mu_0, 0) = \tilde{\chi}_3^2(\mu_0, \mu_1, 0), \tag{50}$$
$$\tilde{\chi}_3^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_1^2(\mu_0, 0, \mu_1) = \tilde{\chi}_3^2(0, \mu_0, \mu_1), \tag{51}$$
$$\tilde{\chi}_4^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_1^2(0, \mu_0, \mu_1) = \tilde{\chi}_2^2(\mu_0, 0, \mu_1), \tag{52}$$
$$\tilde{\chi}_1^1(\mu_0, \mu_1) \;:=\; \tilde{\chi}_2^2(0, \mu_0, \mu_1) = \tilde{\chi}_3^2(\mu_0, 0, \mu_1). \tag{53}$$



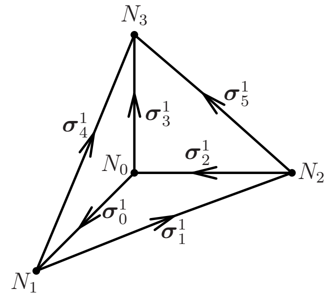


Figure 7: Indexing the entities in the unit triangle and tetrahedron.

Consider a point $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ inside the unit tetrahedron Δ^3_{ref}. The entities which are incident upon N_0 are $\boldsymbol{\sigma}_0^2$, $\boldsymbol{\sigma}_1^2$, $\boldsymbol{\sigma}_3^2$, $\boldsymbol{\sigma}_0^1$, $\boldsymbol{\sigma}_3^1$, $\boldsymbol{\sigma}_2^1$ and $\boldsymbol{\sigma}_0^0$. Hence, we obtain

$$
\begin{aligned}
\mathcal{F}_{\boldsymbol{\sigma}_0^2, N_0}(\Lambda) &= \tilde{\chi}_0^2(1-\lambda_2-\lambda_1, \lambda_2, \lambda_1) = S_0(1-\lambda_2-\lambda_1, \lambda_2, \lambda_1) && (54)\\
\mathcal{F}_{\boldsymbol{\sigma}_1^2, N_0}(\Lambda) &= \tilde{\chi}_1^2(1-\lambda_3-\lambda_1, \lambda_1, \lambda_3) = S_1(1-\lambda_3-\lambda_1, \lambda_1, \lambda_3) && (55)\\
\mathcal{F}_{\boldsymbol{\sigma}_3^2, N_0}(\Lambda) &= \tilde{\chi}_3^2(\lambda_2, 1-\lambda_2-\lambda_3, \lambda_3) = S_3(\lambda_2, 1-\lambda_2-\lambda_3, \lambda_3) && (56)\\
\mathcal{F}_{\boldsymbol{\sigma}_0^1, N_0}(\Lambda) &= \tilde{\chi}_0^1(1-\lambda_1, \lambda_1) = \tilde{\chi}_0^2(1-\lambda_1, 0, \lambda_1) = S_0(1-\lambda_1, 0, \lambda_1) && (57)\\
\mathcal{F}_{\boldsymbol{\sigma}_3^1, N_0}(\Lambda) &= \tilde{\chi}_3^1(1-\lambda_3, \lambda_3) = \tilde{\chi}_1^2(1-\lambda_3, 0, \lambda_3) = S_1(1-\lambda_3, 0, \lambda_3) && (58)\\
\mathcal{F}_{\boldsymbol{\sigma}_2^1, N_0}(\Lambda) &= \tilde{\chi}_2^1(\lambda_2, 1-\lambda_2) = \tilde{\chi}_0^2(1-\lambda_2, \lambda_2, 0) = S_0(1-\lambda_2, \lambda_2, 0) && (59)\\
\mathcal{F}_{\boldsymbol{\sigma}_0^0, N_0}(\Lambda) &= \tilde{\chi}_0^2(1, 0, 0) && (60)
\end{aligned}
$$

By doing the same computation with the entities incident upon N_1, N_2 , N_3, we obtain eventually from (36) the following expression where we take $d = 3$

$$
\begin{aligned}
\mathcal{T}(\Lambda) \;:=\;& \lambda_0 \{S_0(1-\lambda_2-\lambda_1, \lambda_2, \lambda_1) + S_1(1-\lambda_3-\lambda_1, \lambda_1, \lambda_3) + && (61)\\
& S_3(\lambda_2, 1-\lambda_2-\lambda_3, \lambda_3) - S_0(1-\lambda_1, 0, \lambda_1) - S_2(1-\lambda_3, 0, \lambda_3) && (62)\\
& -S_0(1-\lambda_2, \lambda_2, 0) + S_0(1, 0, 0)\} && (63)\\
+\;& \lambda_1 \{S_0(\lambda_0, \lambda_2, 1-\lambda_0-\lambda_2) + S_1(\lambda_0, 1-\lambda_0-\lambda_3, \lambda_3) + && (64)\\
& S_2(1-\lambda_2-\lambda_3, \lambda_2, \lambda_3) - S_1(\lambda_0, 1-\lambda_0, 0) - S_2(1-\lambda_2, \lambda_2, 0) && (65)\\
& -S_1(0, 1-\lambda_3, \lambda_3) + S_1(0, 1, 0)\} && (66)\\
+\;& \lambda_2 \{S_0(\lambda_0, 1-\lambda_0-\lambda_1, \lambda_1) + S_2(\lambda_1, 1-\lambda_1-\lambda_3, \lambda_3) + && (67)\\
& S_3(1-\lambda_0-\lambda_3, \lambda_0, \lambda_3) - S_2(\lambda_1, 1-\lambda_1, 0) - S_3(1-\lambda_0, \lambda_0, 0) && (68)\\
& -S_2(0, 1-\lambda_3, \lambda_3) + S_2(0, 1, 0)\} && (69)\\
+\;& \lambda_3 \{S_1(\lambda_0, \lambda_1, 1-\lambda_0-\lambda_1) + S_2(\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2) + && (70)\\
& S_3(\lambda_2, \lambda_0, 1-\lambda_2-\lambda_0) - S_1(\lambda_0, 0, 1-\lambda_0) - S_1(0, \lambda_1, 1-\lambda_1) && (71)\\
& -S_2(0, \lambda_2, 1-\lambda_2) + S_3(0, 0, 1)\}. && (72)
\end{aligned}
$$

Up to some reordering of the indexation, this very lengthy formula for tetrahedra coincides [15] with that of A. Perronnet in 1998 who did not treat the multidimensional case and who did not present any compact way to formalize his expressions. For the sake of verification, by using the compatibility condition (45), we deduce for $\Lambda = (0, \lambda_1, \lambda_2, \lambda_3)$ that

$$
\begin{aligned}
\mathcal{T}(\Lambda) &= (\lambda_1+\lambda_2+\lambda_3) S_2(1-\lambda 2-\lambda_3, \lambda_2, \lambda_3) = S_2(\lambda_1, \lambda_2, \lambda_3). && (73)\\
\mathcal{T}(1, 0, 0, 0) &= S_1(1, 0, 0) + S_3(0, 1, 0) - S_0(1, 0, 0) = S_1(1, 0, 0) = N_0. && (74)
\end{aligned}
$$

Similar results can be verified for the other three cases about λ_1, λ_2, and λ_3.

3.3.2 Triangular transfinite interpolation

Consider a triangle having nodes $[N_0, N_1, N_2]$. The enumeration of the nodes for the edges by using CGNS enumeration as illustrated in Fig. 7(a) is

$$\boldsymbol{\sigma}_0^1 \longrightarrow [N_0, N_1] \qquad \boldsymbol{\sigma}_1^1 \longrightarrow [N_0, N_3] \qquad \boldsymbol{\sigma}_2^1 \longrightarrow [N_1, N_2]. \tag{75}$$

We are given three functions C_0, C_1, C_2 defined on the unit interval $[0, 1]$ which fulfil the compatibility conditions

$$C_0(0) = C_2(1) = N_0, \qquad C_0(1) = C_1(0) = N_1, \qquad C_1(1) = C_2(0) = N_2. \tag{76}$$

We can repeat the above computation but we state only the final result below. For a point $\Lambda = (\lambda_0, \lambda_1, \lambda_2)$ inside the unit triangle Δ^2_{ref}, relation (36) where $d = 2$ becomes

$$\begin{aligned} \mathcal{T}(\Lambda) \;&:=\; \lambda_0 \left\{C_0(\lambda_1) + C_2(1 - \lambda_2) - C_2(1)\right\} \\ &+\; \lambda_1 \left\{C_1(\lambda_2) + C_0(1 - \lambda_0) - C_0(1)\right\} \\ &+\; \lambda_2 \left\{C_2(\lambda_0) + C_1(1 - \lambda_1) - C_1(1)\right\}. \end{aligned}$$

We can check that $\mathcal{T}(1, 0, 0) = C_0(0) = N_0$, $\mathcal{T}(0, 1, 0) = C_1(0) = N_1$, and $\mathcal{T}(0, 0, 1) = C_1(1) = N_2$. On the other hand, for $\lambda_1 + \lambda_2 = 1$, we have $\mathcal{T}(0, \lambda_1, \lambda_2) = C_1(\lambda_2)$. Similarly, $\mathcal{T}(\lambda_0, 0, \lambda_2) = C_2(\lambda_0)$ and $\mathcal{T}(\lambda_0, \lambda_1, 0) = C_0(\lambda_1)$.

4 Interpolation for multidimensional hypercube

Now that we have insight about multidimensional simplices, we want to turn our attention to the case of hypercubes. Thus, we will introduce first our topological formula by means of appropriately chosen blending functions. Then, we will prove coincidence of that formula to the usual tensor product one.

4.1 Revisiting the 2D Coons patches

Before treating the multidimensional case, let us observe the following relation between the usual 2D Coons patch and a summation relative to the subcubes as illustrated in Fig. 8. By defining

$$\begin{aligned} b_1(u, v) &:= f_0(u) f_0(v), \qquad & b_2(u, v) &:= f_1(u) f_0(v), \\ b_3(u, v) &:= f_1(u) f_1(v), \qquad & b_4(u, v) &:= f_0(u) f_1(v), \end{aligned} \tag{77}$$

we can show (see next Lemma) that they form a set of barycentric blending functions as specified in (P1) (P2) and (P3) of relation (19).
We want to investigate the relation between our construction and the usual transfinite interpolation in Coons form. Let us define

$$\begin{aligned} X(u, v) \;&:=\; b_1(u, v)[(-1)^2\alpha(u) + (-1)^2\delta(v) + (-1)^1\tilde{A}] \\ &+\; b_2(u, v)[(-1)^2\alpha(u) + (-1)^2\beta(v) + (-1)^1\tilde{B}] \\ &+\; b_3(u, v)[(-1)^2\beta(v) + (-1)^2\gamma(u) + (-1)^1\tilde{C}] \\ &+\; b_4(u, v)[(-1)^2\gamma(u) + (-1)^2\delta(v) + (-1)^1\tilde{D}]. \end{aligned} \tag{78}$$

After developing this formula by using (77), we obtain

$$\begin{aligned} X(u, v) \;=\; & \alpha(u) f_0(v) + \delta(u) f_0(u) + \beta(v) f_1(u) + \gamma(u) f_1(v) - \\ & f_0(u) f_0(v)\tilde{A} - f_1(u) f_0(v)\tilde{B} - f_1(u) f_0(v)\tilde{B} - f_1(u) f_1(v)\tilde{C}. \end{aligned}$$

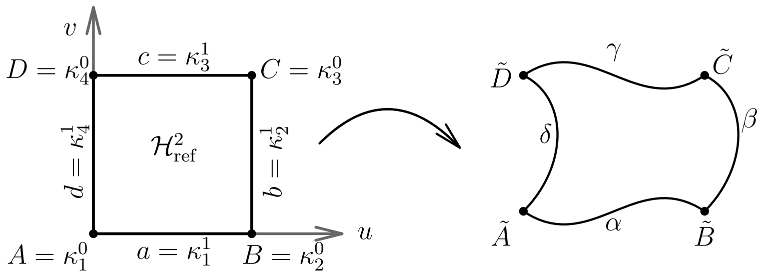
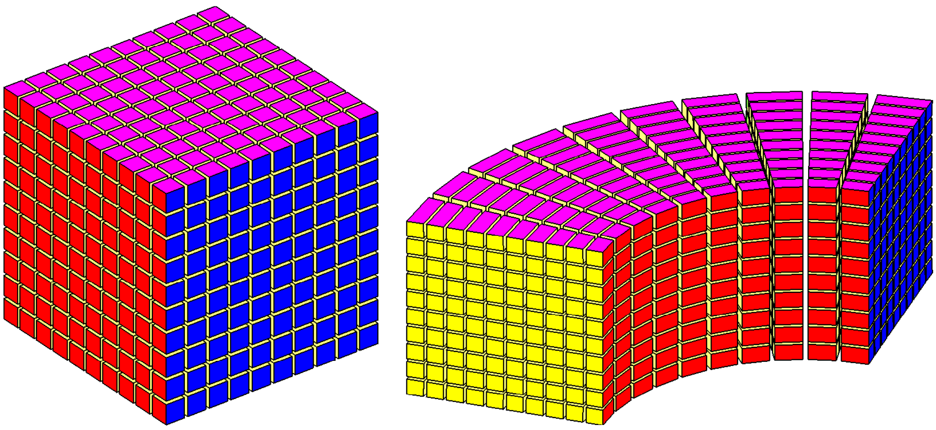


Figure 8: Transfinite interpolation on the 2D hypercube $\mathcal{H}^2_{\text{ref}}$.



(a) (b)

Figure 9: (a)Hexahedralization of the unit cube, (b)Image of the left hexahedralization by a transfinite interpolation.

which coincides with the Coons map in relation (6). The first term in relation (78) corresponds to the node A which is contained by the 1-subcubes $\boldsymbol{\kappa}_1^1 = a$, $\boldsymbol{\kappa}_4^1 = d$ and the 0-subcube $\boldsymbol{\kappa}_1^0$.

4.2 Barycentric Coordinates in Hypercubes

As opposed to the 2D-case, there are many nodes in the multidimensional case. We need an appropriate way of enumerating those nodes. As a consequence, we consider the finite sets

$$\mathcal{I}_k \quad := \quad \left\{\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \mathbb{N}^k : \quad \alpha_i \in \{0,1\}\right\}, \tag{79}$$

$$\mathcal{I}_k^j \quad := \quad \{\boldsymbol{\alpha} \in \mathcal{I}_k : \quad \alpha_j = 0\} \qquad \text{for} \quad j \in [\![1, k]\!]. \tag{80}$$

The nodes of the hypercube $\mathcal{H}^k_{\text{ref}}$ will be denoted by $A^k_{\boldsymbol{\alpha}}$ which have coordinates $A^k_{\boldsymbol{\alpha}} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^k$ to facilitate the presentation of the analysis using tensor product. Define $s_0(t) := 1 - t$ and $s_1(t) := t$. The following lemma determines the barycentric coordinates of a point $\mathbf{u}$ inside the unit hypercube $\mathcal{H}^k_{\text{ref}}$. For the 2D case, the Lemma is illustrated by Fig. 10 where the barycentric coordinates are identified by the areas of the four shaded regions.

Lemma 4.1 *The quantities* $\Lambda(\mathbf{u}) = (\lambda_{\boldsymbol{\alpha}}(\mathbf{u}))_{\boldsymbol{\alpha}\in\mathcal{I}_k}$ *where*

$$\lambda_{\boldsymbol{\alpha}}(\mathbf{u}) := \prod_{p=1}^{k} s_{\alpha_p}(u_p) \tag{81}$$

form a set of barycentric coordinates relative to $\mathcal{I}_k$ *for* $\mathbf{u} = (u_1, ..., u_k) \in \mathcal{H}^k_{\text{ref}}$. *Additionally, the following functions generate a set of barycentric blending functions with respect to* $\Lambda = (\lambda_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathcal{I}_k}$*:*

$$b_{\boldsymbol{\alpha}}(\Lambda) := \prod_{i=1}^{k} f_{\alpha_i}(u_i). \tag{82}$$

PROOF.
The unit sum $\sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} \lambda_{\boldsymbol{\alpha}}(\mathbf{u}) = 1$ is directly shown by induction in which we use

$$\sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} \lambda_{\boldsymbol{\alpha}}(\mathbf{u}) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} \prod_{p=1}^{k} s_{\alpha_p}(u_p) = \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k) \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{k-1}} \prod_{p=1}^{k-1} s_{\alpha_p}(u_p) = 1. \tag{83}$$

We need now to show that $\sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} \lambda_{\boldsymbol{\alpha}}(\mathbf{u}) A^k_{\boldsymbol{\alpha}} = \mathbf{u}$. Thus, by induction with respect to k, we will show the following identity

$$W := \sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} \prod_{p=1}^{k} s_{\alpha_p}(u_p)(\alpha_1, ..., \alpha_k) = (u_1, ..., u_k). \tag{84}$$

It is true for $k = 1$ because $\sum_{\alpha_1=0}^{1} s_{\alpha_1}(u_1)\alpha_1 = s_1(u_1) = u_1$. As hypothesis of induction, we suppose that (84) is correct for $k-1$ and let us show it for k.

$$\begin{aligned}
W &= \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k) \sum_{(\alpha_1,...,\alpha_{k-1})\in\mathcal{I}_k} \Big(\prod_{p=1}^{k-1} s_{\alpha_p}(u_p)\Big)(\alpha_1, ..., \alpha_k) \\
&= \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k) \left(u_1, ..., u_{k-1}, \alpha_n \sum_{(\alpha_1,...,\alpha_{k-1})\in\mathcal{I}_{k-1}} \prod_{p=1}^{k-1} s_{\alpha_p}(u_p) \right) \\
&= \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k) \left(u_1, ..., u_{k-1}, \alpha_n \sum_{(\alpha_1,...,\alpha_{k-1})\in\mathcal{I}_{k-1}} \lambda_{(\alpha_1,...,\alpha_{k-1})} \right) \\
&= \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k)(u_1, ..., u_{k-1}, \alpha_k) \\
&= \left(u_1 \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k), ..., u_{k-1} \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k), \sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k)\alpha_k \right).
\end{aligned}$$

Since $\sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k) = u_k + (1-u_k) = 1$, it disappears in the first $k-1$ coordinates of W above. For the last coordinate, we use the relation $\sum_{\alpha_k=0}^{1} s_{\alpha_k}(u_k)\alpha_k = s_1(u_k) = u_k$ which gives the identity in (84).
As for the existence and uniqueness from (8), we obtain

$$s_0(u_1) \sum_{(\alpha_2,...,\alpha_d)\in\mathcal{I}_{d-1}} \prod_{p=2}^{d} s_{\alpha_p}(u_p) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}^1_d} \prod_{p=1}^{d} s_{\alpha_p}(u_p) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}^1_d} \lambda_{\boldsymbol{\alpha}}(\mathbf{u}) \tag{85}$$

Since $\sum_{(\alpha_2,...,\alpha_d)\in\mathcal{I}_{d-1}}\prod_{p=2}^{d} s_{\alpha_p}(u_p) = 1$, we obtain

$$s_0(u_1) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_d^1} \lambda_{\boldsymbol{\alpha}}(\mathbf{u}). \tag{86}$$

By doing the same thing for u_k we have $s_0(u_k) := \sum_{\boldsymbol{\alpha}\in\mathcal{I}_d^k} \lambda_{\boldsymbol{\alpha}}$ and the value of u_k is obtained by

$$u_k = 1 - s_0(u_k) = 1 - \sum_{\boldsymbol{\alpha}\in\mathcal{I}_d^k} \lambda_{\boldsymbol{\alpha}}. \tag{87}$$

To prove (82), we need to verify the properties (P1), (P2), (P3) of relation (19). In fact, (P1) and (P2) can be easily obtained by using the definition of s_i. Property (P3) can be proved by induction where we use

$$\sum_{\boldsymbol{\alpha}\in\mathcal{I}_k} b_{\boldsymbol{\alpha}}(\Lambda(u_1,...,u_k)) = (f_0(u_k)+f_1(u_k)) \sum_{\boldsymbol{\alpha}\in\mathcal{I}_{k-1}} b_{\boldsymbol{\alpha}}(\Lambda(u_1,...,u_{d-1})), \tag{88}$$

and the fact that f_0 and f_1 sum to unity.

Q.E.D.

4.3 Topological interpolation for $\mathcal{H}_{\text{ref}}^d$

In this section, we will refer to a point $\mathbf{u} \in \mathcal{H}_{\text{ref}}^d$ by using only its barycentric coordinates. We will propose a transfinite interpolation formula on $\mathcal{H}_{\text{ref}}^d$ which makes use exclusively of the barycentric coordinates Λ. As a counterpart of χ_i^k, ψ_i^k and $\tilde{\psi}_i^k$ for the case of simplices from formulae (26), (25) and (28), we would like to introduce now the functions $\psi_{\boldsymbol{\alpha}}^k$, $\chi_{\boldsymbol{\alpha}}^k$ and $\tilde{\psi}_{\boldsymbol{\alpha}}^k$ for the case of hypercube. Consider a subcube $\boldsymbol{\kappa}_{\boldsymbol{\alpha}}^k$ of $\mathcal{H}_{\text{ref}}^d$ where $\boldsymbol{\alpha} \in \mathcal{J}^k$, denote by $\xi_{\boldsymbol{\alpha}}^k$ and $\phi_{\boldsymbol{\alpha}}^k$ the quantities that we introduced in (15) so that

$$\boldsymbol{\kappa}_{\boldsymbol{\alpha}}^k = \mathcal{R}(\xi_{\boldsymbol{\alpha}}^k, \phi_{\boldsymbol{\alpha}}^k) \cap \mathcal{H}_{\text{ref}}^d. \tag{89}$$

Since each element $\mathbf{u} \in \boldsymbol{\kappa}_{\boldsymbol{\alpha}}^k$ is in $\mathcal{H}_{\text{ref}}^k$, we can express it as $\mathbf{u} = \sum_{\boldsymbol{\delta}\in\mathcal{I}_d} \lambda_{\boldsymbol{\delta}} A_{\boldsymbol{\delta}}^d$. We want to determine the set $\mathcal{P}_{\boldsymbol{\alpha}}^k$ of indices $\boldsymbol{\delta}$ such that $\lambda_{\boldsymbol{\delta}} = 0$ for all $\mathbf{u} \in \boldsymbol{\kappa}_{\boldsymbol{\alpha}}^k$. Define

$$\mathcal{P}_{\boldsymbol{\alpha}}^k := \{\boldsymbol{\delta} = (\delta_1,...,\delta_d) \in \mathcal{I}_d : \quad \exists\, i \in \xi_{\boldsymbol{\alpha}}^k \quad \text{with} \quad \delta_i \neq \phi_{\boldsymbol{\alpha}}^k(i)\}. \tag{90}$$

To show that $\lambda_{\boldsymbol{\delta}} = 0$ for all $\boldsymbol{\delta} \in \mathcal{P}_{\boldsymbol{\alpha}}^k$, you note only that $s_{\delta_i}(u_i)$ is equal either to $s_0(1)$ or $s_1(0)$ which are in both cases zero. Because $\lambda_{\boldsymbol{\delta}}$ is the product of all $s_{\delta_j}(u_j)$ as specified in (81), we have $\lambda_{\boldsymbol{\delta}} = 0$ for $\boldsymbol{\delta} \in \mathcal{P}_{\boldsymbol{\alpha}}^k$. Thus, only the nodes $A_{\boldsymbol{\gamma}}^d$ is relevant for $\boldsymbol{\gamma} \notin \mathcal{P}_{\boldsymbol{\alpha}}^k$. Now, we generate an enumerating function $\psi_{\boldsymbol{\alpha}}^k : \mathcal{I}_k \to \mathcal{I}_d \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k$ as follows. Consider an arbitrary but fixed bijection

$$s_{\boldsymbol{\alpha}}^k : [\![1,d]\!] \setminus \xi_{\boldsymbol{\alpha}}^k \to [\![1,k]\!] \tag{91}$$

which is always possible because we have two finite sets having the same cardinality as specified in (16). For $\boldsymbol{\delta} = (\delta_1,...,\delta_k) \in \mathcal{I}_k$, define $\boldsymbol{\gamma} = \psi_{\boldsymbol{\alpha}}^k(\boldsymbol{\delta}) \in \mathcal{I}_d \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k$ to be $\boldsymbol{\gamma} = (\gamma_1,...,\gamma_d)$ such that

$$\gamma_i := \phi_{\boldsymbol{\alpha}}^k(i) \quad \text{for} \quad i \in \xi_{\boldsymbol{\alpha}}^k \tag{92}$$

$$\gamma_i := \delta_{s_{\boldsymbol{\alpha}}^k(i)} \quad \text{for} \quad i \in [\![1,d]\!] \setminus \xi_{\boldsymbol{\alpha}}^k. \tag{93}$$

Thus, we obtain $\boldsymbol{\kappa}_{\boldsymbol{\alpha}}^k = \text{Conv}[A_{\psi_{\boldsymbol{\alpha}}^k(\boldsymbol{\delta})}^d : \quad \boldsymbol{\delta} \in \mathcal{I}_k]$. For every k, we have

$$\mathcal{H}_{\text{ref}}^k = \text{Conv}\{A_{\boldsymbol{\alpha}}^k : \quad \boldsymbol{\alpha} \in \mathcal{I}_k\}. \tag{94}$$

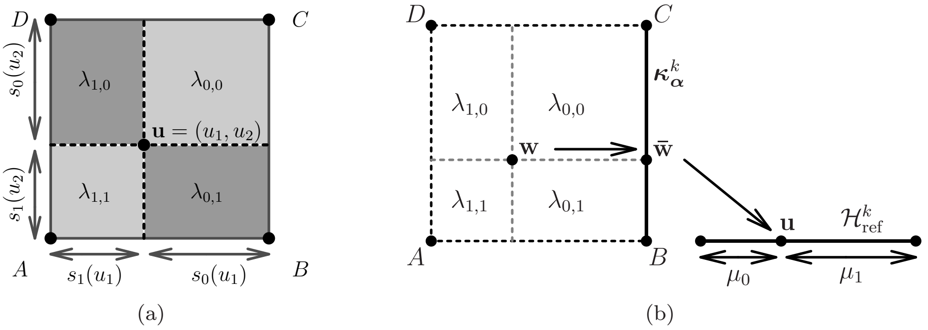


Figure 10: (a)The four shaded areas determine the barycentric coordinates $\lambda_{i,j} = s_i(u_1)s_j(u_2)$ of $\mathbf{u} = (u_1, u_2)$ with respect to $[A, B, C, D]$, (b)The point $\bar{\omega}$ is the projection of $\mathbf{w}$ on the subcube $\boldsymbol{\kappa}^k_{\boldsymbol{\alpha}}$. The new barycentric coordinates are $\mu_0 := \lambda_{1,1} + \lambda_{0,1}$ and $\mu_1 := \lambda_{1,0} + \lambda_{0,0}$.

We can then introduce the function $\chi^k_{\boldsymbol{\alpha}} : \mathcal{H}^k_{\text{ref}} \to \boldsymbol{\kappa}^k_{\boldsymbol{\alpha}}$ transforming $\mathbf{u} = \sum_{\boldsymbol{\delta}\in\mathcal{I}_k} \lambda_{\boldsymbol{\delta}} A^k_{\boldsymbol{\delta}} \in \mathcal{H}^k_{\text{ref}}$ into

$$\chi^k_{\boldsymbol{\alpha}}(\mathbf{u}) = \sum_{\boldsymbol{\delta}\in\mathcal{I}_k} \lambda_{\boldsymbol{\delta}} A^d_{\psi^k_{\boldsymbol{\alpha}}(\boldsymbol{\delta})} \in \boldsymbol{\kappa}^k_{\boldsymbol{\alpha}} \subset \mathcal{H}^d_{\text{ref}}. \tag{95}$$

That means, $\chi^k_{\boldsymbol{\alpha}}$ transforms the vertex $A^k_{\boldsymbol{\delta}}$ of $\mathcal{H}^k_{\text{ref}}$ into $A^d_{\psi^k_{\boldsymbol{\alpha}}(\boldsymbol{\delta})} \in \boldsymbol{\kappa}^k_{\boldsymbol{\alpha}} \subset \mathcal{H}^d_{\text{ref}}$. It is easy to see that $\chi^k_{\boldsymbol{\alpha}}$ is invertible and $(\chi^k_{\boldsymbol{\alpha}})^{-1}$ transforms $A^d_{\psi^k_{\boldsymbol{\alpha}}(\boldsymbol{\delta})}$ to $A^k_{\boldsymbol{\delta}}$.
As we did for the simplex case, we are given functions defined on $\mathcal{H}^{d-1}_{\text{ref}}$ of number $2d$:

$$\mathcal{W}^{d-1}_{\boldsymbol{\alpha}} : \mathcal{H}^{d-1}_{\text{ref}} \to \mathbb{R}^d \quad \text{where} \quad \widetilde{\boldsymbol{\kappa}}^{d-1}_{\boldsymbol{\alpha}} := \mathcal{W}^{d-1}_{\boldsymbol{\alpha}}(\mathcal{H}^{d-1}_{\text{ref}}). \tag{96}$$

Let us denote by B the union of the images $\widetilde{\boldsymbol{\kappa}}^{d-1}_{\boldsymbol{\alpha}}$ and our objective is to find a function on $\mathcal{H}^d_{\text{ref}}$ such that it transforms $\partial\mathcal{H}^d_{\text{ref}}$ into B. From the input functions $\mathcal{W}^{d-1}_{\boldsymbol{\alpha}}$, let us now construct a function $\mathcal{Z}$ which is defined on the boundary $\partial\mathcal{H}^d_{\text{ref}}$. First, note that the boundary is the union of the cubical faces:

$$\partial\mathcal{H}^d_{\text{ref}} = \bigcup_{\boldsymbol{\alpha}\in\mathcal{J}^{d-1}} \boldsymbol{\kappa}^{d-1}_{\boldsymbol{\alpha}}. \tag{97}$$

Consider a $\mathbf{w} \in \partial\mathcal{H}^d_{\text{ref}}$ and let us introduce its image $\mathcal{Z}(\mathbf{w})$ as follows. Due to relation (97), there exists $\boldsymbol{\alpha} \in \mathcal{J}^{d-1}$ such that $\mathbf{w} \in \boldsymbol{\kappa}^{d-1}_{\boldsymbol{\alpha}}$. As a consequence, we have the representation $\mathbf{w} = \sum_{\boldsymbol{\delta}\in\mathcal{I}^{d-1}} \lambda_{\boldsymbol{\delta}} A_{\psi^{d-1}_{\boldsymbol{\alpha}}(\boldsymbol{\delta})}$ as specified in (95). From that, we can introduce $\mathbf{u} \in \mathcal{H}^{d-1}_{\text{ref}}$ as $\mathbf{u} := \sum_{\boldsymbol{\delta}\in\mathcal{I}^{d-1}} \lambda_{\boldsymbol{\delta}} A^{d-1}_{\boldsymbol{\delta}}$ and we define

$$\mathcal{Z}(\mathbf{w}) := \mathcal{W}^{d-1}_{\boldsymbol{\alpha}}(\mathbf{u}). \tag{98}$$

By using the function $\mathcal{Z}$, we want to deduce some mapping $\tilde{\chi}^h_{\boldsymbol{\beta}}$ for all subcubes of lower dimension. For that, we need the canonical injection $\mathcal{N}$ such that $\forall\,\mathbf{u} \in \boldsymbol{\kappa}^h_{\boldsymbol{\beta}}$ we have $\mathcal{N}(\mathbf{u}) := \mathbf{u} \in \partial\mathcal{H}^d_{\text{ref}}$. The induced function is defined as

$$\tilde{\chi}^h_{\boldsymbol{\beta}} := \mathcal{Z} \circ \mathcal{N} \circ \chi^h_{\boldsymbol{\beta}} \tag{99}$$

which is illustrated in Fig. 11.
As we have done in (31), for $\boldsymbol{\kappa} := \boldsymbol{\kappa}^k_{\boldsymbol{\alpha}}$ which contains the node $A^d_{\boldsymbol{\beta}}$ we use the mapping $\tilde{\chi}^k_{\boldsymbol{\alpha}} : \mathcal{H}^k_{\text{ref}} \to \widetilde{\boldsymbol{\kappa}}^k_{\boldsymbol{\alpha}}$. We want to introduce now the function $\pi^k_{\boldsymbol{\alpha}}$ which transforms a point $\mathbf{w} \in \mathcal{H}^d_{\text{ref}}$ with barycentric coordinates $\Lambda = (\lambda_{\boldsymbol{\delta}})_{\boldsymbol{\delta}\in\mathcal{I}_d}$ to a point

$$\begin{tikzcd}
\partial\mathcal{H}^d_{\mathrm{ref}} \subset \mathcal{H}^d_{\mathrm{ref}} \subset \mathbb{R}^d \arrow[rr, "\mathcal{Z}"] & & B := \bigcup_{\boldsymbol{\alpha}\in\mathcal{J}^{d-1}} \widetilde{\boldsymbol{\kappa}}^{d-1}_{\boldsymbol{\alpha}} \\
\boldsymbol{\kappa}^h_{\boldsymbol{\beta}} \arrow[u, "\mathcal{N}"] & \mathcal{H}^h_{\mathrm{ref}} \subset \mathbb{R}^h \arrow[l, "\chi^h_{\boldsymbol{\beta}}"'] \arrow[r, "\tilde{\chi}^h_{\boldsymbol{\beta}}"] & \widetilde{\boldsymbol{\kappa}}^h_{\boldsymbol{\beta}}
\end{tikzcd}$$

Figure 11: Inducing the mapping $\tilde{\chi}^h_{\boldsymbol{\beta}}$ from $\tilde{\chi}^k_{\boldsymbol{\alpha}}$

$\mathbf{u} \in \mathcal{H}^k_{\mathrm{ref}}$ with barycentric coordinates $\boldsymbol{\mu} = (\mu_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_k}$ as illustrated in Fig. 10(b). For a point $\mathbf{w} \in \mathcal{H}^d_{\mathrm{ref}}$ where we have $\mathbf{w} = \sum_{\boldsymbol{\delta}\in\mathcal{I}_d} \lambda_{\boldsymbol{\delta}} A^d_{\boldsymbol{\delta}}$, we want now to define a projection $\bar{\mathbf{w}} \in \boldsymbol{\kappa}^k_{\boldsymbol{\alpha}}$. In order to obtain $\bar{\mathbf{w}} = \sum_{\boldsymbol{\delta}\in\mathcal{I}_d} \bar{\lambda}_{\boldsymbol{\delta}} A^d_{\boldsymbol{\delta}}$, we define its barycentric coordinates $\bar{\lambda}_{\boldsymbol{\delta}}$ as follows. For $\boldsymbol{\delta} \in \mathcal{I}_d$, we define

$$\mathcal{Y}_{\boldsymbol{\delta}} = \mathcal{Y}_{\boldsymbol{\delta},k,\boldsymbol{\alpha}} := \{\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_d) \in \mathcal{I}_d : \quad \gamma_j = \delta_j \quad \text{for each} \quad j \notin \xi^k_{\boldsymbol{\alpha}}\} \tag{100}$$

the index of non-fixed component in $\boldsymbol{\kappa}^k_{\boldsymbol{\alpha}}$ and

$$\bar{\lambda}_{\boldsymbol{\delta}} := K_{\boldsymbol{\delta}} \sum_{\boldsymbol{\gamma}\in\mathcal{Y}_{\boldsymbol{\delta}}} \lambda_{\boldsymbol{\gamma}} \qquad \text{where} \qquad K_{\boldsymbol{\delta}} := \prod_{p\in\xi^k_{\boldsymbol{\alpha}}} s_{\delta_p}(\phi^k_{\boldsymbol{\alpha}}(p)). \tag{101}$$

Note that $K_{\boldsymbol{\delta}} \in \{0, 1\}$ where it is zero if $\boldsymbol{\delta} \in \mathcal{P}^k_{\boldsymbol{\alpha}}$. That means

$$\bar{\mathbf{w}} = \sum_{\boldsymbol{\delta}\in\mathcal{I}_d\setminus\mathcal{P}^k_{\boldsymbol{\alpha}}} \bar{\lambda}_{\boldsymbol{\delta}} A^d_{\boldsymbol{\delta}}. \tag{102}$$

From this, we obtain a point $\mathbf{u} \in \mathcal{H}^k_{\mathrm{ref}}$ as illustrated in Fig. 10(b). We have

$$\mathbf{u} = \pi^k_{\boldsymbol{\alpha}}(\mathbf{w}) := \sum_{\boldsymbol{\gamma}\in\mathcal{I}_k} \mu_{\boldsymbol{\gamma}} A^k_{\boldsymbol{\gamma}}. \tag{103}$$

where $\mu_{\boldsymbol{\gamma}} := \bar{\lambda}_{(\psi^k_{\boldsymbol{\alpha}})^{-1}(\boldsymbol{\gamma})}$. We define therefore

$$\mathcal{F}_{\boldsymbol{\kappa},A_{\boldsymbol{\beta}}}(\Lambda) := \tilde{\chi}^k_{\boldsymbol{\alpha}} \circ \pi^k_{\boldsymbol{\alpha}}(\Lambda) = \tilde{\chi}^k_{\boldsymbol{\alpha}}(\boldsymbol{\mu}) \qquad \text{where} \qquad \boldsymbol{\mu} = (\mu_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\mathcal{I}_k}. \tag{104}$$

The topological expression of transfinite interpolation with respect to $\mathcal{H}^d_{\mathrm{ref}}$ is defined as

$$\mathcal{T}(\Lambda) := (-1)^{d+1} \sum_{\boldsymbol{\alpha}\in\mathcal{I}_d} b_{\boldsymbol{\alpha}}(\Lambda) \sum_{\boldsymbol{\kappa}\in S_{\boldsymbol{\alpha}}} (-1)^{\dim(\boldsymbol{\kappa})} \mathcal{F}_{\boldsymbol{\kappa},A_{\boldsymbol{\alpha}}}(\Lambda) \tag{105}$$

where $S_{\boldsymbol{\alpha}}$ is the set of all subcubes $\boldsymbol{\kappa}$ containing the corner $A_{\boldsymbol{\alpha}}$.

4.4 Developing Tensor Product Interpolation

Let us recall the usual notion of multidimensional tensor product transfinite interpolation. Consider two functions $f_0, f_1 : [0,1] \to \mathbb{R}$ such that $f_0(t) + f_1(t) = 1$ and $f_i(j) = \delta_{i,j}$. Let $\mathcal{C}([0,1]^d, \mathbb{R}^d)$ be the space of continuous functions from $[0,1]^d$ to $\mathbb{R}^d$. An operator $\mathcal{P}_q$ is defined from $\mathcal{C}([0,1]^d, \mathbb{R}^d)$ to itself as follows:

$$\begin{aligned} \mathcal{P}_q(\mathbf{x})(u_1, ..., u_d) \quad := \quad & f_1(u_p)\mathbf{x}(u_1, ..., u_{p-1}, 1, u_{p+1}, ..., u_d) + \\ & f_0(u_p)\mathbf{x}(u_1, ..., u_{p-1}, 0, u_{p+1}, ..., u_d). \end{aligned}$$

For two such operators $\mathcal{P}_p$ and $\mathcal{P}_q$, introduce the Boolean sum

$$(\mathcal{P}_p \oplus \mathcal{P}_q)(\mathbf{x}) := \mathcal{P}_p(\mathbf{x}) + \mathcal{P}_q(\mathbf{x}) - \mathcal{P}_p(\mathcal{P}_q(\mathbf{x})). \tag{106}$$

The transfinite interpolation of hypercube is

$$\bigoplus_{i=1}^{n} \mathcal{P}_i(\mathbf{x}). \tag{107}$$

The Boolean sum character of a Coons patch has been discovered by W. Gordon. In fact, we can prove the following theorem similarly as we did for Theorem 3.3. But here we would like to use an alternative way because we want to display that our formula coincides with the usual tensor product case.
Denote by $\mathcal{D}_n$ the set of nonempty subsets of $\{1, ..., n\}$. For any $\xi \in \mathcal{D}_n$ and $\boldsymbol{\alpha} \in \mathcal{I}_n$, we define

$$\mathcal{Q}_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(u_1, ..., u_d) := \mathbf{x}(v_1, ..., v_d), \tag{108}$$

here $v_i := \alpha_i$=constant if $i \in \xi$ and $v_i := u_i$ otherwise.

Lemma 4.2 *Consider the d-dimensional hypercube $\mathcal{H}^d_{\mathrm{ref}}$ and a function $\mathbf{x} \in \mathcal{C}(\mathcal{H}^d_{\mathrm{ref}}, \mathbb{R}^m)$. Then, for any $n \leq d$, we have the following equality:*

$$\left[\bigoplus_{i=1}^{n} \mathcal{P}_i(\mathbf{x})\right](\mathbf{u}) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_n} \left[\left(\prod_{i=1}^{n} f_{\alpha_i}(u_i)\right) \sum_{\xi\in\mathcal{D}_n} (-1)^{\mathrm{card}(\xi)} \mathcal{Q}_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(\mathbf{u}) \right], \tag{109}$$

where $\mathbf{u} = (u_1, ..., u_d)$.

PROOF.
Denote by $A_n(\mathbf{x}) := \bigoplus_{i=1}^{n} \mathcal{P}_i(\mathbf{x})$. Thus, we will show by induction on n that

$$A_n(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_n} \left[\left(\prod_{i=1}^{n} f_{\alpha_i}(u_i)\right) \sum_{\xi\in\mathcal{D}_n} (-1)^{\mathrm{card}(\xi)} \mathcal{Q}_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(\mathbf{u}) \right]. \tag{110}$$

For the case $n = 1$, we deduce from definition that

$$\begin{aligned} A_1(\mathbf{x})(\mathbf{u}) &= \mathcal{P}_1(\mathbf{x})(\mathbf{u}) = f_1(u_1)\mathbf{x}(1, u_2, ..., u_d) + f_0(u_1)\mathbf{x}(0, u_2, ..., u_d) \\ &= \textstyle\sum_{\alpha_1=0}^{1} f_{\alpha_1}(u_1)\mathbf{x}(\alpha_1, u_2, ..., u_d) \\ &= \textstyle\sum_{\alpha_1=0}^{1} f_{\alpha_1}(u_1)\mathcal{Q}_{(\xi:=\{1\},\boldsymbol{\alpha}:=\{\alpha_1\})}\mathbf{x}(u_1, u_2, ..., u_d). \end{aligned} \tag{111}$$

As hypothesis of induction, we suppose (110) for $n-1$. Denote by $\mathcal{D}^k_n$ the set of elements ξ of $\mathcal{D}_n$ such that $\mathrm{card}(\xi) = k$. We can decompose $\mathcal{D}^k_n$ into three partitions:

$$\mathcal{D}^k_n = \mathcal{D}^k_{n-1} \cup \{n\} \cup \mathcal{M}^k_n \tag{112}$$

where $\mathcal{M}^k_n := \{\bar{\xi} := \xi \cup \{n\} \quad \text{with} \quad \xi \in \mathcal{D}^{k-1}_{n-1}\}$ for $k = 2, ..., n$ and $\mathcal{M}^1_n := \emptyset$. By the Boolean sum property (106), we have

$$A_n(\mathbf{x}) = A_{n-1}(\mathbf{x}) + \mathcal{P}_n(\mathbf{x}) - A_{n-1}(\mathcal{P}(\mathbf{x})). \tag{113}$$

Since $f_1(u_d) + f_0(u_d) = 1$, we have $A_{n-1}(\mathbf{x}) = f_1(u_d)A_{n-1} + f_0(u_d)A_{n-1} = \sum_{\alpha_d=0}^{1} f_{\alpha_d}A_{n-1}$. Hence, for the first term of (113), we have

$$\begin{aligned} A_{n-1} &= \sum_{\alpha_d=0}^{1} f_{\alpha_d} \sum_{(\alpha_1,\ldots,\alpha_{d-1})\in\mathcal{I}_{n-1}} \left(\prod_{i=1}^{n-1} f_{\alpha_i}(u_i)\right) \sum_{k=1}^{n-1} \sum_{\xi\in\mathcal{D}^k_{n-1}} (-1)^k \mathcal{Q}_{(\xi,(\alpha_1,\ldots,\alpha_{d-1}))}(\mathbf{x})(\mathbf{u}) \\ &= \sum_{\boldsymbol{\alpha}:=(\alpha_1,\ldots,\alpha_d)\in\mathcal{I}_n} \left(\prod_{i=1}^{n} f_{\alpha_i}(u_i)\right) \sum_{k=1}^{n-1} \sum_{\xi\in\mathcal{D}^k_{n-1}} (-1)^k \mathcal{Q}_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(\mathbf{u}) \end{aligned}$$

On the other hand, for the second term of (113) , as we have done in relation (111), we obtain

$$\begin{aligned} \mathcal{P}_d(\mathbf{x}) &= f_1(u_d)\mathbf{x}(u_1,...,u_{d-1},1) + f_0(u_d)\mathbf{x}(u_1,...,u_{d-1},0) &(114)\\ &= \sum_{\boldsymbol{\alpha}\in\mathcal{I}_d}\prod_{i=1}^{d} f_{\alpha_i}(u_i)\mathcal{Q}_{(\xi:=\{d\},\boldsymbol{\alpha})}(\mathbf{x})(u_1,...,u_{d-1},u_d) &(115)\end{aligned}$$

Finally, by using (114), we obtain for any $\boldsymbol{\alpha}\in\mathcal{I}_{d-1}$ that ($\boldsymbol{\alpha}$ does not contain d):

$$\begin{aligned} Q_{\xi,\boldsymbol{\alpha}}(\mathcal{P}_d(\mathbf{x})) &= f_1(u_d)Q_{(\xi,\boldsymbol{\alpha})}(\mathcal{P}_d(\mathbf{x}))(u_1,...,u_{d-1},1) + f_0(u_d)Q_{(\xi,\boldsymbol{\alpha})}(\mathcal{P}_d(\mathbf{x}))(u_1,...,u_{d-1},0)\\ &= f_1(u_d)Q_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(u_1,...,u_{d-1},1) + f_0(u_d)Q_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(u_1,...,u_{d-1},0)\\ &= \sum_{\alpha_d=0}^{1} f_{\alpha_d}(u_d)Q_{(\xi,\boldsymbol{\alpha})}(\mathbf{x})(u_1,...,u_{d-1},\alpha_d)\\ &= \sum_{\alpha_d=0}^{1} f_{\alpha_d}(u_d)Q_{(\bar{\xi}:=\xi\cup\{d\},\boldsymbol{\alpha})}(\mathbf{x})(u_1,...,u_{d-1},u_d)\end{aligned}$$

As a consequence,

$$\begin{aligned} A_{d-1}(\mathcal{P}_d(\mathbf{x})) &= \sum_{(\alpha_1,...,\alpha_{d-1})\in\mathcal{I}_{d-1}}\left(\prod_{i=1}^{d-1} f_{\alpha_i}(u_i)\right)\sum_{k=1}^{d-1}\sum_{\xi\in\mathcal{D}_{d-1}^k}(-1)^k Q_{(\xi,(\alpha_1,...,\alpha_{d-1}))}(P(x))(\mathbf{u})\\ &= \sum_{(\alpha_1,...,\alpha_d)\in\mathcal{I}_d}\left(\prod_{i=1}^{d} f_{\alpha_i}(u_i)\right)\sum_{k=1}^{d-1}\sum_{\xi\in\mathcal{D}_{d-1}^k}(-1)^k Q_{(\bar{\xi},(\alpha_1,...,\alpha_{d-1}))}(x)(\mathbf{u})\\ &= \sum_{(\alpha_1,...,\alpha_d)\in\mathcal{I}_d}\left(\prod_{i=1}^{d} f_{\alpha_i}(u_i)\right)\sum_{k=2}^{d}\sum_{\xi\in\mathcal{D}_{d-1}^k}(-1)^{k+1} Q_{(\bar{\xi},(\alpha_1,...,\alpha_{d-1}))}(x)(\mathbf{u})\\ &= -\sum_{(\alpha_1,...,\alpha_d)\in\mathcal{I}_d}\left(\prod_{i=1}^{d} f_{\alpha_i}(u_i)\right)\sum_{k=2}^{d}\sum_{\bar{\xi}\in M_d^k}(-1)^k Q_{(\bar{\xi},(\alpha_1,...,\alpha_{d-1}))}(x)(\mathbf{u})\end{aligned}$$

Q.E.D.

4.5 Coincidence of the two representations

In this section, we would like to express the relation of the formula in (105) and the tensor product formula $\bigoplus_{i=1}^{n}\mathcal{P}_i(\mathbf{x})$ in the multidimensional case. Before displaying the results, let us consider the following property of the previously induced function $\tilde{\chi}^h_{\boldsymbol{\beta}}$.

Proposition 4.3 *For $i\notin\xi^k_{\boldsymbol{\alpha}}$ we have*

$$\mathcal{I}_d^i = \bigcup_{\boldsymbol{\delta}\in\mathcal{I}_d^i\setminus\mathcal{P}^k_{\boldsymbol{\alpha}}}\mathcal{Y}_{\boldsymbol{\delta}} \qquad (116)$$

so that for $\Lambda=(\lambda_{\boldsymbol{\delta}})_{\boldsymbol{\delta}\in\mathcal{I}_d}$ defining a point $\mathbf{w}=(w_1,...,w_d)\in\mathcal{H}^d_{\mathrm{ref}}$, the barycentric coordinates $\bar{\Lambda}=(\bar{\lambda}_{\boldsymbol{\delta}})_{\boldsymbol{\delta}\in\mathcal{I}_d}$ given by (101) specifies a point $\bar{\mathbf{w}}=(w_1,...,w_d)$ such that

$$\bar{w}_i = w_i \qquad \text{for} \qquad i\notin\xi^k_{\boldsymbol{\alpha}}. \qquad (117)$$

PROOF.
Consider a $\boldsymbol{\gamma} \in \mathcal{Y}_{\boldsymbol{\delta}}$ for some $\boldsymbol{\delta} \in \mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k$. That is, $\gamma_j = \delta_j$ for each $j \notin \xi_{\boldsymbol{\alpha}}^k$. Since $i \notin \xi_{\boldsymbol{\alpha}}^k$, we have $\gamma_i = \delta_i = 0$. That means, $\boldsymbol{\gamma} \in \mathcal{I}_d^i$. That implies $\cup_{\boldsymbol{\delta} \in \mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k} \mathcal{Y}_{\boldsymbol{\delta}} \subset \mathcal{I}_d^i$. On the other hand, consider $\boldsymbol{\gamma} \in \mathcal{I}_d^i = (\mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k) \cup \mathcal{P}_{\boldsymbol{\alpha}}^k)$. In the case that $\boldsymbol{\gamma} \in \mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k$, we simply define $\boldsymbol{\delta} := \boldsymbol{\gamma}$ and obtain that $\boldsymbol{\gamma} \in \mathcal{Y}_{\boldsymbol{\delta}}$. In the latter case where $\boldsymbol{\gamma} \in \mathcal{P}_{\boldsymbol{\alpha}}^k$, we define $\boldsymbol{\delta} \in \mathcal{I}_d^i$ such that

$$\begin{array}{rcll} \delta_j & := & \gamma_j & \text{for} \quad j \notin \xi_{\boldsymbol{\alpha}}^k \\ \delta_j & := & \phi_{\boldsymbol{\alpha}}^k(j) & \text{for} \quad j \in \xi_{\boldsymbol{\alpha}}^k \end{array}$$

By construction, $\boldsymbol{\delta} \notin \mathcal{P}_{\boldsymbol{\alpha}}^k$ so that $\boldsymbol{\delta} \in \mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k$ implying $\boldsymbol{\gamma} \in \mathcal{Y}_{\boldsymbol{\delta}}$. In both cases, we obtain $\mathcal{I}_d^i \subset \cup_{\boldsymbol{\delta} \in \mathcal{I}_d^i \setminus \mathcal{P}_{\boldsymbol{\alpha}}^k} \mathcal{Y}_{\boldsymbol{\delta}}$. The union (116) follows from the above two inclusions. In order to show (117), we use (87) and we note that $\bar{\lambda}_{\boldsymbol{\delta}} = 0$ for $\boldsymbol{\delta} \in \mathcal{P}_{\boldsymbol{\alpha}}^k$, so that we obtain

$$\bar{w}_i = 1 - \sum_{\boldsymbol{\delta} \in \mathcal{I}_d^i} \bar{\lambda}_{\boldsymbol{\delta}} = 1 - \sum_{\substack{\boldsymbol{\delta} \in \mathcal{I}_d^i \\ \boldsymbol{\delta} \notin \mathcal{P}_{\boldsymbol{\alpha}}^k}} \bar{\lambda}_{\boldsymbol{\delta}} = 1 - \sum_{\substack{\boldsymbol{\delta} \in \mathcal{I}_d^i \\ \boldsymbol{\delta} \notin \mathcal{P}_{\boldsymbol{\alpha}}^k}} \sum_{\boldsymbol{\gamma} \in \mathcal{Y}_{\boldsymbol{\delta}}} \lambda_{\boldsymbol{\gamma}} \tag{118}$$

By using (116), we deduce $\bar{w}_i = w_i$ for $i \notin \xi_{\boldsymbol{\alpha}}^k$.

Q.E.D.

Proposition 4.4 *Consider any $\boldsymbol{\kappa}_{\boldsymbol{\beta}}^d$ which is any subcube of $\mathcal{H}_{\text{ref}}^k$. There exists a certain subset $\xi \subset [\![1, d]\!]$ with* $\text{Card}(\xi) = d - h$ *such that for $\mathbf{u} = (u_1, ..., u_h) \in \mathcal{H}_{\text{ref}}^h$, its image by $\tilde{\chi}_{\boldsymbol{\beta}}^h$ is $\mathcal{Z}(\bar{\mathbf{w}})$ for some $\bar{\mathbf{w}} = (\bar{w}_1, ..., \bar{w}_d) \in \mathcal{H}_{\text{ref}}^d$ where*

$$\left\{ \begin{array}{lcll} \bar{w}_i & \in & \{0, 1\} & \text{for} \quad i \in \xi, \\ \bar{w}_i & \in & \{u_1, ..., u_h\} & \text{for} \quad i \in [\![1, d]\!] \setminus \xi. \end{array} \right. \tag{119}$$

PROOF.
As seen in (81), any $\mathbf{u} \in \mathcal{H}_{\text{ref}}^h$ can be expressed in terms of $A_{\boldsymbol{\delta}}^h$ as $\mathbf{u} = \sum_{\boldsymbol{\delta} \in \mathcal{I}_h} \bar{\lambda}_{\boldsymbol{\delta}} A_{\boldsymbol{\delta}}^h$. As a consequence, from the definition of $\chi_{\boldsymbol{\beta}}^h$ in (95), we obtain

$$\bar{\mathbf{w}} := \chi_{\boldsymbol{\beta}}^h(\mathbf{u}) = \sum_{\boldsymbol{\delta} \in \mathcal{I}_h} \bar{\lambda}_{\boldsymbol{\delta}} A_{\psi_{\boldsymbol{\beta}}^h(\boldsymbol{\delta})}^d \in \boldsymbol{\kappa}_{\boldsymbol{\beta}}^h \subset \mathcal{H}_{\text{ref}}^d \tag{120}$$

which is unchanged by the canonical injection such as $\bar{\mathbf{w}} = \mathcal{N}(\bar{\mathbf{w}})$. Define now the barycentric coordinates $(\mu_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma} \in \mathcal{I}^d}$ as follows:

$$\left\{ \begin{array}{lcll} \mu_{\boldsymbol{\gamma}} & := & \bar{\lambda}_{(\psi_{\boldsymbol{\beta}}^h)^{-1}(\boldsymbol{\gamma})} & \text{if } \boldsymbol{\gamma} \in \mathcal{I}_d \setminus \mathcal{P}_{\boldsymbol{\beta}}^h, \\ \mu_{\boldsymbol{\gamma}} & := & 0 & \text{if } \boldsymbol{\gamma} \in \mathcal{P}_{\boldsymbol{\beta}}^h. \end{array} \right. \tag{121}$$

As a consequence, by the invertibility of $\psi_{\boldsymbol{\beta}}^h : \mathcal{I}^h \to \mathcal{I}^d \setminus \mathcal{P}_{\boldsymbol{\beta}}^h$, we obtain

$$\bar{\mathbf{w}} = \sum_{\boldsymbol{\delta} \in \mathcal{I}_h} \bar{\lambda}_{\boldsymbol{\delta}} A_{\psi_{\boldsymbol{\beta}}^h(\boldsymbol{\delta})}^d = \sum_{\boldsymbol{\gamma} \in \mathcal{I}_d \setminus \mathcal{P}_{\boldsymbol{\beta}}^h} \bar{\lambda}_{(\psi_{\boldsymbol{\beta}}^h)^{-1}(\boldsymbol{\gamma})} A_{\boldsymbol{\gamma}}^d = \sum_{\boldsymbol{\gamma} \in \mathcal{I}_d} \mu_{\boldsymbol{\gamma}} A_{\boldsymbol{\gamma}}^d. \tag{122}$$

The fact that $\bar{\mathbf{w}}$ belongs to $\boldsymbol{\kappa}_{\boldsymbol{\beta}}^h$ implies that for each $i \in \xi_{\boldsymbol{\beta}}^h$ we have $\bar{w}_i = \phi_{\boldsymbol{\beta}}^h(i) \in \{0, 1\}$ as stated in (90). That proves the first relation in (121). Consider now an index $i \in [\![1, d]\!] \setminus \xi_{\boldsymbol{\beta}}^h$. As proved in (87), we have

$$\bar{w}_i = 1 - \sum_{\boldsymbol{\gamma} \in \mathcal{I}_d^i} \mu_{\boldsymbol{\gamma}}. \tag{123}$$

Due to the property of $s := s^h_{\boldsymbol\beta}$ in (91), we have $\boldsymbol\delta \in \mathcal{I}_h^{s(i)}$ if $\boldsymbol\gamma = \psi^h_{\boldsymbol\beta}(\boldsymbol\delta)$ belongs to $\mathcal{I}^i_d$. As a result, we obtain

$$\bar w_i = 1 - \sum_{\boldsymbol\delta\in\mathcal{I}_h^{s(i)}} \mu_{\psi^h_{\boldsymbol\beta}(\boldsymbol\delta)} = 1 - \sum_{\boldsymbol\delta\in\mathcal{I}_h^{s(i)}} \bar\lambda_{\boldsymbol\delta} = u_{s(i)} \quad \forall\, i \in [\![1,d]\!] \setminus \xi^h_{\boldsymbol\delta} \tag{124}$$

which demonstrates the second relation of (121). The set to be sought is therefore $\xi := \xi^h_{\boldsymbol\beta}$ which has cardinality $d-h$ as specified in (16).

Q.E.D.

Theorem 4.5 *By using the barycentric coordinates in (81) and the barycentric blending functions in (82) for the d-dimensional hypercube $\mathcal{H}^d_{\text{ref}}$, the function*

$$\mathcal{T}(\Lambda) := (-1)^{d+1} \sum_{\boldsymbol\alpha\in\mathcal{I}_d} b_{\boldsymbol\alpha}(\Lambda) \sum_{\boldsymbol\kappa\in S_{\boldsymbol\alpha}} (-1)^{\dim(\boldsymbol\kappa)} \mathcal{F}_{\boldsymbol\kappa,A_{\boldsymbol\alpha}}(\Lambda) \tag{125}$$

is a transfinite interpolation. That is, $\mathcal{T}(\Lambda)$ verifies for each $q = 0, ..., d$:

$$\mathcal{T}(\Lambda^{\boldsymbol\alpha}) = A_{\boldsymbol\alpha} \tag{126}$$
$$\mathcal{T}(\lambda_0, ..., \lambda_{q-1}, 0, \lambda_{q+1}, ..., \lambda_d) = \chi^{d-1}_q(\lambda_{\psi^{d-1}_q(1)}, ..., \lambda_{\psi^{d-1}_q(d)}). \tag{127}$$

PROOF.
To prove this theorem, we need to show the equality of $\mathcal{T}(\Lambda)$ and the tensor product relation (109). Consider any $\xi \in \mathcal{D}_d$ and $\boldsymbol\delta \in \mathcal{I}_d$ and we let $k := d - \text{Card}(\xi)$. Define a function $\phi : \xi \to \{0,1\}$ by $\phi(i) = \delta_i$. Thus, as shown in (16), we can consider the subcube $\boldsymbol\kappa$ of $\mathcal{H}^d_{\text{ref}}$ with respect to $\boldsymbol\alpha := (\xi, \phi)$. That is,

$$\xi^k_{\boldsymbol\alpha} = \xi \qquad \text{and} \qquad \phi^k_{\boldsymbol\alpha} = \phi. \tag{128}$$

As a consequence, $\boldsymbol\kappa$ is a subcube of dimension k. Since $\dim(\boldsymbol\kappa) = d - \text{card}(\xi)$, we have $(-1)^{\text{card}(\xi)} = (-1)^d(-1)^{\dim(\boldsymbol\kappa)}$. Thus, there is a correspondence between $(\xi, \boldsymbol\delta) \in \mathcal{D}_d \times \mathcal{I}_d$ and a subcube $\boldsymbol\kappa$ of dimension k. Since $k = \dim(\boldsymbol\kappa) = d - \text{Card}(\xi)$, we have

$$(-1)^{\text{Card}(\xi)} = (-1)^d(-1)^{\dim(\boldsymbol\kappa)}. \tag{129}$$

By combining Proposition 4.3 and Proposition 4.4, we have that for any $\mathbf{w} \in \mathcal{H}^d_{\text{ref}}$ having barycentric coordinates Λ, the point $\mathbf{u} = \Pi^k_{\boldsymbol\alpha}(\mathbf{w})$ defined in (103) maps by $\chi^k_{\boldsymbol\alpha}$ to $\bar{\mathbf{w}}$ such that $w_i = \bar w_i$ for $i \notin \xi^k_{\boldsymbol\alpha}$. Additionally, we have

$$\mathbf{x}(\bar w_1, ..., \bar w_d) = \mathcal{Q}_{(\xi^k_{\boldsymbol\alpha},\boldsymbol\delta)}\mathbf{x}(w_1, ..., w_d). \tag{130}$$

Since $\tilde\chi^k_{\boldsymbol\alpha}$ is the composition of $\chi^k_{\boldsymbol\alpha}$ and $\mathcal{Z}$, we have

$$\mathcal{F}_{\boldsymbol\kappa,A_{\boldsymbol\beta}}(\Lambda) = \mathcal{Q}_{(\xi^k_{\boldsymbol\alpha},\boldsymbol\delta)}\mathbf{x}(w_1, ..., w_d) \tag{131}$$

From Lemma 4.2, we obtain

$$[A_d(\mathbf{x})](\mathbf{u}) = \sum_{\boldsymbol\delta\in\mathcal{I}_d} \left[(\prod_{i=1}^d f_{\delta_i}(u_i)) \sum_{\xi\in\mathcal{D}_d} (-1)^{\text{card}(\xi)} \mathcal{Q}_{(\xi,\boldsymbol\delta)}(\mathbf{x})(\mathbf{u}) \right], \tag{132}$$

By using $b_{\boldsymbol\delta}(\Lambda) = \prod_{i=1}^d f_{\delta_i}(u_i)$, relation (128) and (129), we obtain

$$[A_d(\mathbf{x})](\mathbf{u}) = (-1)^d \sum_{\boldsymbol\delta\in\mathcal{I}_d} b_{\boldsymbol\delta}(\Lambda) \sum_{\boldsymbol\kappa\in S_{\boldsymbol\delta}} (-1)^{\dim(\boldsymbol\kappa)} \mathcal{F}_{\boldsymbol\kappa,A_{\boldsymbol\beta}}(\Lambda) \tag{133}$$

from which we deduce (125).

Q.E.D.

5 Discussion

In the opinion of the author, the method here can be applied to any polytope of arbitrary dimension as long as it is convex. In that case, the barycentric coordinates can be introduced with respect to the corners of the polytope and one can introduce also blending functions. For the case of pentahedron, a formula of Perronnet [15] confirms that conjecture. Since treating that is beyond the scope of this paper, we will consider that problem in a future discussion.

We believe that the previous theory can be very helpful for the progress of geometric method for the preparation of data which are needed for multiscale approaches. Methods based upon refineable structures [13] are usually very efficient in practice [12] because they give rise to subdivision algorithms which can be used for the construction of multiscale bases [5]. Such a multilevel setting produces in general good accuracy with low computational cost [5]. The rate between cost and accuracy has been demonstrated to be optimal [5] as specified by N-term approximation.

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