Institut für Numerische Simulation
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Computer lab
Numerical Methods for Thin Elastic Sheets

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Problem sheet 3 **June 3rd, 2013**

For this lab session, we want to compute the basis functions of the Discrete Kirchhoff Triangle (DKT) for solving the variational form of the discrete Kirchhoff plate equation given by

$$a(w_h, \phi_h) = a(\theta_h(w_h), \psi_h(\phi_h)) = (f, \phi_h),$$

where

$$\begin{aligned} a(w_h, \phi_h) &:= \int_\omega \frac{E\delta^3}{12(1-\nu^2)} \left(\begin{pmatrix} w_{h,11} \\ w_{h,22} \\ w_{h,12} + w_{h,21} \end{pmatrix}^T \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \phi_{h,11} \\ \phi_{h,22} \\ \phi_{h,12} + \phi_{h,21} \end{pmatrix} \right) \mathrm{d}x \\ &= \int_\omega \frac{E\delta^3}{12(1-\nu^2)} \left(\begin{pmatrix} \theta^1_{h,1} \\ \theta^2_{h,2} \\ \theta^1_{h,2} + \theta^2_{h,1} \end{pmatrix}^T \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \psi^1_{h,1} \\ \psi^2_{h,2} \\ \psi^1_{h,2} + \psi^2_{h,1} \end{pmatrix} \right) \mathrm{d}x \\ &=: a(\theta_h(w_h), \psi_h(\phi_h)), \quad w_h, \phi_h \in \mathcal{W}_h, \theta_h, \psi_h \in \Theta_h. \text{ (For details see lecture notes.)} \end{aligned}$$

The following conditions determine the DKT element:

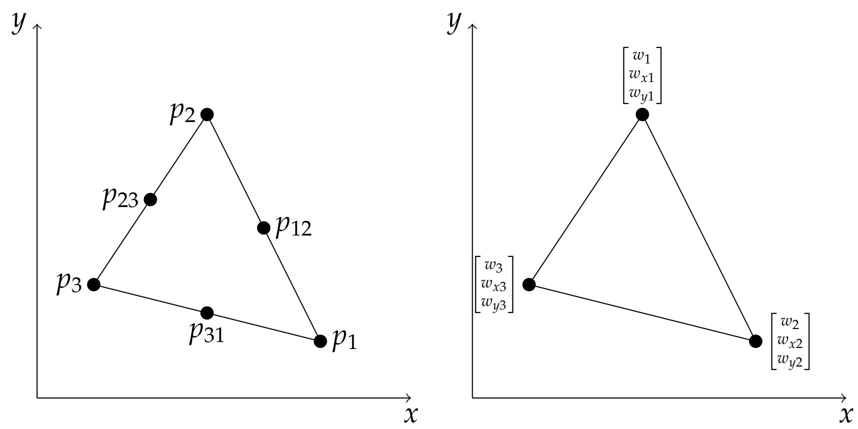

Figure 1: Left: Degrees of freedom of the $\mathcal{P}_2$ element. Right: Degrees of freedom of the DKT element.

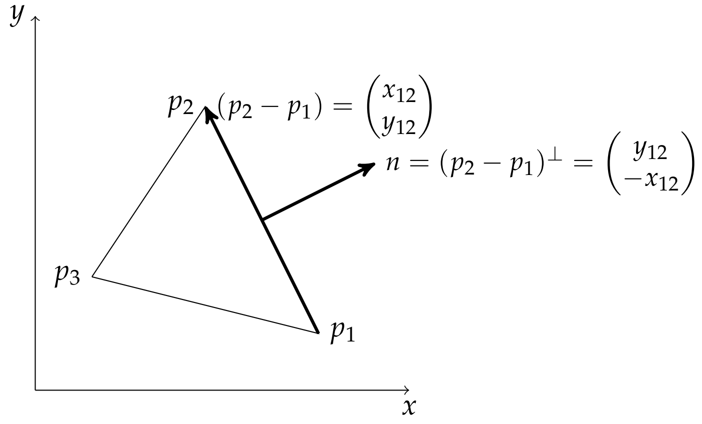

Figure 2: Tangential direction $(p_2 - p_1)$ and normal direction $(p_2 - p_1)^\perp$

(i) $\mathcal{W}_h := \{w_h \in H_0^1(\omega) \mid \nabla w_h$ is continuous at all nodes of $\mathcal{T}_h,\ w_h|_T \in \mathcal{P}_{3,\text{red}}\ \forall\ T \in \mathcal{T}_h\}$,

(ii) $\Theta_h := \{\theta_h \in (H_0^1(\omega))^2 \mid \theta_h|_T \in (\mathcal{P}_2)^2$ and $\theta_h \cdot n \in \mathcal{P}_1(E)\ \forall\ T \in \mathcal{T}_h,\ \forall E \in \mathcal{E}(T)\}$,

(iii) $\theta_h(w_h)(p_i) = \nabla w_h(p_i),\ i = 1, 2, 3$,

(iv) $\theta_h(w_h)(p_{ij})(p_j - p_i) = \nabla w_h(p_{ij})(p_j - p_i),\ p_{ij} = \frac{1}{2}(p_j + p_i)$.

Here, $p_i = \begin{pmatrix} x_i & y_i \end{pmatrix}^T$ ($i = 1, 2, 3$) denotes a corner point of triangle T and $x_{ij} := x_j - x_i$ as well as $y_{ij} := y_j - y_i$.

Task: Derive the basis functions in the degrees of freedom vector

$$U^T = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & w_2 & w_{x2} & w_{y2} & w_3 & w_{x3} & w_{y3} \end{bmatrix},$$

i.e., determine vectors H_x and H_y such that

$$\theta_h(w_h)(\xi, \eta) = \begin{pmatrix} H_x(\xi, \eta)^T U \\ H_y(\xi, \eta)^T U \end{pmatrix}.$$

Hint: To determine the rhs of condition (iv) consider an edge parametrized by a path $\gamma(s) := p_i + s(p_j - p_i)$ and make yourself clear how $w_h(\gamma(s))$ and $\frac{d}{ds} w_h(\gamma(s))$ can be expressed in terms of the U. Furthermore, consider $\theta_h(w_h) \cdot n$ and make yourself clear, how this can be written in terms of ∇w_h.