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Numerical Methods for Thin Elastic Sheets

Summer term 2013

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Problem sheet 5

June 17th, 2013

We have

$$\mathcal{W}_h := \{w_h \in H_0^1(\omega) \mid \nabla w_h \text{ is continuous at all nodes of } \mathcal{T}_h,\ w_h|_T \in \mathcal{P}_{3,\text{red}}\ \forall\ T \in \mathcal{T}_h\}$$
$$\Theta_h := \{\theta_h \in (H_0^1(\omega))^2 \mid \theta_h|_T \in (\mathcal{P}_2)^2 \text{ and } \theta_h \cdot n \in \mathcal{P}_1(E)\ \forall\ T \in \mathcal{T}_h,\ \forall E \in \mathcal{E}(T)\}$$

and want to minimize $\mathcal{E}$ over $\mathcal{W}_h$, where

$$\mathcal{E}[w] = \frac{E\delta^3}{24(1-\nu^2)} \int_\omega \left((w_{,11} + w_{,22})^2 + 2(1-\nu)(w_{,12}^2 - w_{,11}w_{,22}) \right) - \delta f \cdot w \, \mathrm{d}x \,.$$

Using the Kirchhof condition $\nabla w = \theta$ (i.e. $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$) we want to solve

$$< \mathcal{E}'[w], \phi > = 0 \qquad \forall \phi \in \mathcal{W}_h$$
$$\Longleftrightarrow \qquad a\big(\, \theta(w),\, \theta(\phi)\, \big) = \delta (f, \phi)_{L^2(\omega)} \qquad \forall \phi \in \mathcal{W}_h$$

with

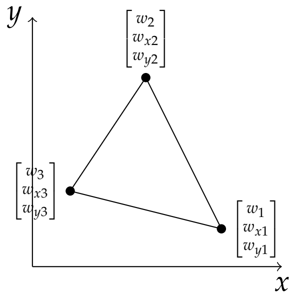
$$a\big(\, \theta(w),\, \theta(\phi)\, \big) = \int_\omega \nabla\theta(w)^T C[E,\nu,\delta]\, \nabla\theta(\phi)\, \mathrm{d}x \,, \qquad C[E,\nu,\delta] := \frac{E\delta^3}{12(1-\nu^2)} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}$$

where we use the notation $\nabla\theta(w) := [\theta(w)_{1,1},\, \theta(w)_{2,2},\, \theta(w)_{1,2} + \theta_{2,1}]^T$.

Now that we can evaluate $\nabla\theta$ at quadrature points we are able to set up the stiffness matrix $A \in \mathbb{R}^{3n,3n}$, where $n := |\mathcal{N}_h|$, with $A_{ij} = a\big(\, \theta_i,\, \theta_j\, \big)$ for basis functions θ_i of Θ_h and solve the linear system $A\bar{w} = \bar{f}$.

Note that $\bar{w} \in \mathbb{R}^{3n}$ as we have three degrees of freedom at each vertex $x_i \in \mathcal{N}_h$, namely $w(x_i), w_{,1}(x_i)$ and $w_{,2}(x_i)$.

When assembling the corresponding FE operator, it is crucial that the nine local degrees of freedom (cf. figure) are mapped correctly to their global positions in A. As the function values $w(x_i)$ at each vertex $x_i \in \mathcal{N}_h$ define naturally the first n DOFs (i.e. $w(x_i) \mapsto i$), we recomment to extend the mapping by $w_{,1}(x_i) \mapsto n + i$ and $w_{,2}(x_i) \mapsto 2n + i$.



When solving $A\bar{w} = \bar{f}$ one might want to impose different types of boundary conditions. Prescribing $w(x_i) = 0$ for all $x_i \in \mathcal{N}_h \cap \partial\omega$ is called *simply supported* boundary condition, whereas we refer to a *clamped* boundary condition when we additionally prescribe derivatives at each boundary vertex. The corresponding boundary masks should make use of the local to global mapping described above.

As a first example we will consider a constant load on the right hand side, where

$$f\big|_T = \frac{|T|}{3}[q, 0, 0, q, 0, 0, q, 0, 0]$$

on each triangle T, i.e. $f_i = \frac{q}{3}\sum_{T:x_i \in T} |T|$ for $i = 1, \ldots, n$ ($f_i = 0$ else) and $\bar{f} = (f_i)_i$.

Tasks:

- Complete the configurator `DKTPlateTriangMeshConfigurator<>` for the Discrete Kirchhoff Triangle by using the corresponding template in *labsheetTemplates/labsheet5/DKTFE.h*. In particular, implement the local to global mapping described above and set up boundary masks for clamped and simply supported boundary conditions.
- Implement a FE operator to assemble A. Therefore derive from a suitable FE operator provided in `QuocMesh`, e.g. `aol::FELinMatrixWeightedStiffInterface<>`, such that you only have to evaluate the matrix argument $C[E, \nu, \delta]$ at each quadrature point.
- Complete the main program and test your DKT element with ω being a triangulation of $[0, 1]^2$ and constant loads on the right hand side.