Scientific Computing II

Summer Semester 2014
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Excercise sheet 1.

Closing date **15.04.2014**.

Theoretical exercise 1. (Variant of the Bramble-Hilbert Lemma [5 points])

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected domain with Lipschitz boundary and cone condition and let $d = \sup_{a,b\in\Omega} \|a-b\|_2$. Let $\mathbb{Q}_p := \operatorname{span}\{x^k y^l \mid 0 \leq k,l \leq p\}$ be the space of two-dimensional polynomials of degree at most p in each variable. Assume that for $p \in \mathbb{N}$ there exists a constant $C > 0$ which is independent of d such that

$$\inf_{q\in\mathbb{Q}_p} \|f-q\|_{H^{p+1}(\Omega)} \leq C\,[f]_{H^{p+1}(\Omega)}$$

for all $f \in H^{p+1}(\Omega)$ where

$$[f]_{H^m(\Omega)} := \left(\left\| \frac{\partial^m f}{\partial x^m} \right\|^2_{L_2(\Omega)} + \left\| \frac{\partial^m f}{\partial y^m} \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}.$$

Prove that for $p \geq 2$ there exists a constant $C > 0$ independent of d such that

$$\inf_{q\in\mathbb{Q}_p} \sum_{i=0}^{p+1} d^i |f-q|_{H^i(\Omega)} \leq C d^{p+1}\,[f]_{H^{p+1}(\Omega)}$$

for all $f \in H^{p+1}(\Omega)$.

Theoretical exercise 2. (Inverse estimates [5 points])

Let Ω and d be as in exercise 1. Let $p \in [1,\infty]$, $m \in \mathbb{N}$ and $P \subset W^{m,p}(\Omega)$ a finite-dimensional subspace. Prove that for $0 \leq t \leq s \leq m$ there exists a $C > 0$ such that the inverse estimate

$$|f|_{W^{s,p}} \leq C d^{t-s} |f|_{W^{t,p}}$$

holds for all $f \in P$.

Theoretical exercise 3. ((Local) Quasi-uniformity [5 points])

For a finite element K let h_K be its diameter and ρ_K the diameter of the largest circle that can be inscribed in K. Assume that the corresponding family of triangulations is quasi-uniform. Denote by

$$\tilde{K} := \operatorname{int}(\{\cup K' \mid \bar{K} \cap \bar{K}' \neq \emptyset\})$$

the patch to the element K.

a) Prove that there exists a generic constant $C > 0$ such that

$$h_K \leq C h_{K'} \qquad \text{and} \qquad \frac{h_K}{\rho_K} \leq C \frac{h_{K'}}{\rho_{K'}}$$

for all $K' \in \tilde{K}$.

b) Let $T_K(x) := A_K x + b_K$ be the affine linear transformation from the reference triangle $\hat{K} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$ onto K. Prove that there exists a generic constant $C > 0$ such that

$$\begin{aligned} \|A_K\| &\leq Ch_K \\ \|A_K^{-1}\| &\leq C\rho_K^{-1} \\ C\rho_K^2 &\leq |\det(A_K)| \leq Ch_K^2 \end{aligned}$$

where $\|\cdot\|$ denotes the Euclidean operator norm for matrices.

Theoretical exercise 4. (Lagrangian finite elements [5 points])

Let $(T_h)_h$ be a family of quasi-uniform triangulations of a domain Ω as above with Lagrangian elements. For one element K let the local basis functions be $\phi_1, \ldots, \phi_n$. Show that there exist constants $C, \tilde{C}$ such that

a) $\|\phi_i\|_{L_\infty(\Omega)} \leq C \quad \forall\, i \in \{1, \ldots, n\}$

b) $\|\nabla\phi_i\|_{L_\infty(\Omega)} \leq \tilde{C} \cdot h_K^{-1} \quad \forall\, i \in \{1, \ldots, n\}$