Theoretical exercise 1. (Hierarchical error estimator: Condition Number [10 points])
For a sequence of linear Lagrange element spaces in two dimensions \( V_0 \subset \ldots \subset V_L \) let the hierarchical decomposition be given by
\[
V_l = V_{l-1} \oplus W_l
\]
for all \( l = 1, \ldots, L \), where \( W_l \) is the space of nodal basis functions for which the basis points appeared first on level \( l \), i.e. \( W_l = \text{span}\{\phi_i^{(l)}\}_{i \in B_l} \) with the notation from the lecture. Calculate the order of the condition number for the stiffness matrix
\[
K_{11} = \left( a(\phi_i^{(l)}, \phi_j^{(l)}) \right)_{i,j \in B_l}
\]
of the standard Poisson problem with \( a(u, v) = \int_{[0,1]^2} \nabla u \cdot \nabla v \, dx \) for uniform refinements of the following FE-spaces:

a) Linear triangle elements given by

\[
\begin{array}{c}
V_0 \\
\rightarrow V_1 \\
\rightarrow \ldots
\end{array}
\]

b) Bilinear rectangular elements given by

\[
\begin{array}{c}
V_0 \\
\rightarrow V_1 \\
\rightarrow \ldots
\end{array}
\]

Theoretical exercise 2. (Constant of the strenghntened Cauchy inequality [10 points])
For the examples in this exercise the meshes are the same as in Exercise 1. For \( a(u, v) = \int_{[0,1]^2} \nabla u \cdot \nabla v \, dx \) and the hierarchical enrichment decomposition \( V_l := V_{l-1} \oplus W_l \) where \( W_l := \text{span}\{\phi_i^{(l)} \mid i \in B_l\} \) is the complementary space of basis functions which are new on level \( l \), calculate an as low as possible upper bound for the constant \( \gamma \) from the strengthened Cauchy inequality for both linear triangular as well as bilinear quadratic elements.