Exercise sheet 5.  

Closing date 13.05.2014.

Theoretical exercise 1. (Diameter under bisection for special triangles [5 points])

For the triangle \( \Delta \) with corners \( A, B, C \) let \( AB \geq BC \geq AC \). Let \( D \) denote the midpoint of the edge \( AB \) and \( F \) the midpoint of \( BC \). Assume that \( AC \geq \max(\overline{CD}, \overline{AD}) \) and \( \overline{CD} \geq \overline{CF} \).

Always bisecting triangles along their longest edges we get a family \( \Delta_{n,i}, i = 1, \ldots, 2^n \) for level \( n > 0 \) with \( \Delta_{0,1} = \Delta \).

Prove that after \( 2^n \) bisection steps we have \( 2^{2n-1} \) triangles each with diameter \( \frac{\text{diam}(\Delta)}{2^n} \) and \( 2^{2n-1} \) triangles each with diameter \( \frac{\text{diam}(\Delta)}{2^ {2n-1}} \cdot \frac{\overline{CD}}{\overline{AB}} \).

Theoretical exercise 2. (Diameter under bisection for generic triangles [10 points])

Let \( l_k \) denote the length of the longest edge of triangles on level \( k \). As on sheet 3, we call a triangle \( \Delta_{0,1} \) “suitable” if there exists an \( N > 0 \) such that

\[
l_{2k} \leq \left( \frac{\sqrt{3}}{2} \right)^{\min(k,N)} \cdot \left( \frac{1}{2} \right)^{\max(k-N,0)} \cdot l_0 \quad \text{for } k \geq 0.
\]

Prove that every triangle is suitable.

Hints:

- Assume the set \( S \) of triangles which are not suitable is non-empty. Then with \( t \) from exercise 1 of sheet 3 let \( \bar{t} = \sup_{\tau \in S} t(\tau) \). Now choose a triangle \( \Delta_{0,1} = \Delta(ABC) \in S \) with \( t(\Delta(ABC)) \) close enough to \( \bar{t} \) and show that \( \Delta(ABC) \) has to be suitable which contradicts the assumption that \( S \) was non-empty.

- Let w.l.o.g. \( AB \geq BC \geq AC \). Note that there are now three cases that can happen when bisecting twice to get \( \Delta_{2,i}, i = 1, \ldots, 4 \). This is due to the fact that any side of the triangle \( \Delta(CAD) \) could be bisected. Also take into account triangles on level 3 when bisecting \( \Delta(CDF) \). There only \( CD \) or \( CF \) could be bisected. Combine these cases to get six overall cases, i.e.

  1. \( AC \) in \( \Delta(CAD) \) and \( CD \) in \( \Delta(CDF) \)
  2. \( AC \) in \( \Delta(CAD) \) and \( CF \) in \( \Delta(CDF) \)
  3. \( CD \) in \( \Delta(CAD) \) and \( CD \) in \( \Delta(CDF) \)
  4. \( CD \) in \( \Delta(CAD) \) and \( CF \) in \( \Delta(CDF) \)
  5. \( AD \) in \( \Delta(CAD) \) and \( CD \) in \( \Delta(CDF) \)
  6. \( AD \) in \( \Delta(CAD) \) and \( CF \) in \( \Delta(CDF) \)

Study all six different cases and make use of already proven results in other exercises. Also note that \( \overline{CD}^2 = \frac{\overline{AC}^2}{2} + \frac{\overline{BC}^2}{2} - \frac{\overline{AB}^2}{4} \).
Theoretical exercise 3. (Residual error estimator [5 points])

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected domain with Lipschitz boundary and cone condition. Let $u, u_h$ be the solutions over $V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \}$ and over the FE-space $V_h \subset V$ of the model problem

\begin{equation}
- \Delta u + cu = f \text{ on } \Omega, \\
u = 0 \text{ on } \Gamma_1 \subset \partial \Omega, \\
\frac{\partial u}{\partial n} = g \text{ on } \Gamma_2 = \partial \Omega \setminus \Gamma_1
\end{equation}

transferred into the weak formulation. For the corresponding triangulation $T_h$ let $\Omega_e$ denote the union of all elements in $T_h$ containing the edge $e$. Prove that

$$a(u - u_h, \overline{R}\phi_e) = \int_{\Omega_e} \phi_e r_h \overline{R} dx + \int_{\Gamma_2} \phi_e R_h \overline{R} ds$$

where $a$ is the bilinear form corresponding to the weak form of (1) and $\phi_e$ is the edge bubble function of $e$. Furthermore, $r_h$ and $R_h$ denote the interior and the boundary residuals on each element and $\overline{R}$ is an approximation to the boundary residual from some suitable finite-dimensional space.