Scientific Computing II

Summer Semester 2014
Lecturer: Prof. Dr. Beuchler
Assistent: Bastian Bohn

universitätbonn

Excercise sheet 5.

Closing date **13.05.2014**.

Theoretical exercise 1. (Diameter under bisection for special triangles [5 points])

For the triangle Δ with corners A, B, C let $\overline{AB} \geq \overline{BC} \geq \overline{AC}$. Let D denote the midpoint of the edge AB and F the midpoint of BC. Assume that $\overline{AC} \geq \max(\overline{CD}, \overline{AD})$ and $\overline{CD} \geq \overline{CF}$.

Always bisecting triangles along their longest edges we get a family $\Delta_{n,i}, i = 1, \ldots, 2^n$ for level $n > 0$ with $\Delta_{0,1} = \Delta$.

Prove that after $2n$ bisection steps we have 2^{2n-1} triangles each with diameter $\frac{\mathrm{diam}(\Delta)}{2^n}$ and 2^{2n-1} triangles each with diameter $\frac{\mathrm{diam}(\Delta)}{2^{n-1}} \cdot \frac{\overline{CD}}{\overline{AB}}$.

Theoretical exercise 2. (Diameter under bisection for generic triangles [10 points])

Let l_k denote the length of the longest edge of triangles on level k. As on sheet 3, we call a triangle $\Delta_{0,1}$ "suitable" if there exists an $N > 0$ such that

$$l_{2k} \leq \left(\frac{\sqrt{3}}{2}\right)^{\min(k,N)} \cdot \left(\frac{1}{2}\right)^{\max(k-N,0)} \cdot l_0 \quad \text{for } k \geq 0.$$

Prove that every triangle is suitable.

Hints:

- Assume the set S of triangles which are not suitable is non-empty. Then with t from exercise 1 of sheet 3 let $\hat{t} = \sup_{\tau \in S} t(\tau)$. Now choose a triangle $\Delta_{0,1} = \Delta(ABC) \in S$ with $t(\Delta(ABC))$ close enough to $\hat{t}$ and show that $\Delta(ABC)$ has to be suitable which contradicts the assumption that S was non-empty.

- Let w.l.o.g. $\overline{AB} \geq \overline{BC} \geq \overline{AC}$. Note that there are now three cases that can happen when bisecting twice to get $\Delta_{2,i}, i = 1, \ldots, 4$. This is due to the fact that any side of the triangle $\Delta(CAD)$ could be bisected. Also take into account triangles on level 3 when bisecting $\Delta(CDF)$. There only CD or CF could be bisected. Combine these cases to get six overall cases, i.e.

  1. AC in $\Delta(CAD)$ and CD in $\Delta(CDF)$
  2. AC in $\Delta(CAD)$ and CF in $\Delta(CDF)$
  3. CD in $\Delta(CAD)$ and CD in $\Delta(CDF)$
  4. CD in $\Delta(CAD)$ and CF in $\Delta(CDF)$
  5. AD in $\Delta(CAD)$ and CD in $\Delta(CDF)$
  6. AD in $\Delta(CAD)$ and CF in $\Delta(CDF)$

  Study all six different cases and make use of already proven results in other exercises. Also note that $\overline{CD}^2 = \frac{\overline{AC}^2}{2} + \frac{\overline{BC}^2}{2} - \frac{\overline{AB}^2}{4}$.

Theoretical exercise 3. (Residual error estimator [5 points])

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected domain with Lipschitz boundary and cone condition. Let u, u_h be the solutions over $V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\}$ and over the FE-space $V_h \subset V$ of the model problem

$$\begin{aligned} -\Delta u + cu &= f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1 \subset \partial\Omega, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \Gamma 2 = \partial\Omega \setminus \Gamma_1 \end{aligned} \tag{1}$$

transferred into the weak formulation. For the corresponding triangulation T_h let Ω_e denote the union of all elements in T_h containing the edge e. Prove that

$$a(u - u_h, \overline{R}\phi_e) = \int_{\Omega_e} \phi_e r_h \overline{R} dx + \int_e \phi_e R_h \overline{R} ds$$

where a is the bilinear form corresponding to the weak form of (1) and ϕ_e is the edge bubble function of e. Furthermore, r_h and R_h denote the interior and the boundary residuals on each element and $\overline{R}$ is an approximation to the boundary residual from some suitable finite-dimensional space.