Scientific Computing II

Summer Semester 2014
Lecturer: Prof. Dr. Beuchler
Assistent: Bastian Bohn

universitätbonn

Excercise sheet 8.

Closing date **03.06.2014**.

Theoretical exercise 1. (Upper bound for the BPX decomposition [20 points])

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected domain with Lipschitz boundary and cone condition. Define $a(u,v) := \int_\Omega \nabla u \nabla v dx$. for $u, v \in H_0^1(\Omega)$.
Let $\mathbb{V}^l$ be a sequence of nested piecewise linear FE-spaces with basis ϕ_i^l, $l = 0, \dots, L$ and $i = 1, \dots, N_l$ created by uniformly refining an initial mesh T_0 into meshes T_l. The (largest) meshwidth in T_l is denoted by h_l.
Prove that for the BPX decomposition

$$\mathbb{V}^L := \sum_{l=0}^{L} \sum_{i=1}^{N_l} \mathbb{V}_i^L$$

with $\mathbb{V}_i^L = \operatorname{span}\{\phi_i^l\}$ it holds

$$a(u,u) \leq c \sum_{l=0}^{L} \sum_{i=1}^{N_l} a(u_i^l, u_i^l) \tag{1}$$

for some constant $c > 0$ (independent of L) and $u \in \mathbb{V}^L$, $u_i^l \in \mathbb{V}_i^l$ and $u = \sum_{l=0}^{L} \sum_{i=1}^{N_l} u_i^l$.

To this end, follow the subtasks:

a) Prove that

$$a(u,u) \leq \|\Theta\|_2^2 \sum_{l=0}^{L} \sum_{i=1}^{N_l} a(u_i^l, u_i^l),$$

where $\Theta = \left(\theta_{i,j}^{k,l}\right)_{(k,i),(l,j)}$ is the matrix of all angles

$$\theta_{i,j}^{k,l} := \cos(\sphericalangle \mathbb{V}_i^k, \mathbb{V}_j^l) = \sup_{u \in \mathbb{V}_i^k, v \in \mathbb{V}_j^l} \frac{a(u,v)}{\sqrt{a(u,u)a(v,v)}}.$$

b) Let $\Omega_i^l := \operatorname{supp}(\phi_i^l)$ and let w.l.o.g. $k \leq l$. Prove that $\theta_{i,j}^{k,l} = 0$ if

(i) $\operatorname{int}(\Omega_i^k \cap \Omega_j^l) = \emptyset$ or

(ii) $\Omega_j^l \subset \tau_{i,*}^k$, where $\tau_{i,*}^k$ is one element in the support of ϕ_i^k.

Prove that there exists an $c > 0$ such that in all other cases $\theta_{i,j}^{k,l} \leq c \frac{h_l}{h_k}$.

c) Let $\Theta^{k,l} := \left(\theta_{i,j}^{k,l}\right)_{i=1,\dots,N_k \text{ and } j=1,\dots,N_l}$. Prove the following two estimates:

(i) For $k > l$ it holds $\|\Theta^{k,l}\|_1 \leq c\frac{h_k}{h_l}$.

(ii) For $k \leq l$ it holds $\|\Theta^{k,l}\|_1 \leq c$.

d) Let $\tilde{\Theta} := \left(\|\Theta^{k,l}\|_1 \cdot \left(\frac{h_l}{h_k}\right)^{\frac{1}{2}}\right)_{k,l}$. Prove that there exists a constant $\tilde{c} > 0$ such that

$$\|\tilde{\Theta}\|_1 \leq \tilde{c}.$$

e) Prove that for symmetric, real matrices A any eigenvalue λ fulfills $|\lambda| \leq \|A\|_1$. Prove that for a block matrix $A = (A_{a,b})_{a,b}$ with matrices $A_{a,b}$ it holds $\|A\|_1 \leq \|\bar{A}\|_1$ where $\bar{A} = (\|A_{a,b}\|_1)_{a,b}$.

f) Combine the above proven results to prove (1).