Aufgabe 2: Berechnen Sie die Integrale:

$$\text{a)} \int_0^1 xe^x \, dx \,, \quad \text{b)} \int \frac{x^2}{x^3+5} \, dx \,, \quad \text{c)} \int \sqrt{x} \, dx \,.$$

Tipp: a) mit partieller Integration, b) mit Substitutionsregel, c) mit partieller Integration oder unter Verwendung von $\sqrt{x} = x^{\frac{1}{2}}$.

LÖSUNG:

a)

$$\int_0^1 xe^x \, dx = xe^x|_0^1 - \int_0^1 e^x \, dx = 1 \cdot e^1 - 0 \cdot e^0 - e^x|_0^1 = e - (e-1) = 1.$$

Partielle Integration mit:

$$f(x) = e^x, \; f'(x) = e^x, \; g(x) = x, \; g'(x) = 1.$$

b)

$$\int \frac{x^2}{x^3+5} \, dx = \frac{1}{3} \int \frac{dz}{z} = \frac{1}{3} \log|z| = \frac{1}{3} \log|x^3+5|.$$

Substitution: $z := x^3 + 5 \;\Rightarrow\; dz = 3x^2 \, dx \;\Leftrightarrow\; x^2 \, dx = \frac{1}{3} dz.$

c)

$$\int \sqrt{x} \, dx = \int x^{1/2} \, dx = \frac{x^{1/2+1}}{1/2+1} = \frac{x^{3/2}}{3/2} = \frac{2}{3}\sqrt{x^3} = \frac{2}{3}x\sqrt{x} \quad (x \geq 0!).$$

Oder mit partieller Integration:

$$\begin{aligned} \int \sqrt{x} \, dx &= \int 1 \cdot \sqrt{x} \, dx = x \cdot \sqrt{x} - \int x \cdot \frac{1}{2\sqrt{x}} \, dx = \\ &= x \cdot \sqrt{x} - \frac{1}{2} \int \sqrt{x} \, dx \\ \Rightarrow \frac{3}{2} \int \sqrt{x} \, dx &= x \cdot \sqrt{x} \;\Leftrightarrow\; \int \sqrt{x} \, dx = \frac{2}{3} x \sqrt{x} \,. \end{aligned}$$

Dabei war

$$f(x) = x, \; f'(x) = 1, \; g(x) = \sqrt{x}, \; g'(x) = \frac{1}{2\sqrt{x}}.$$

Aufgabe 3: Berechnen Sie folgende Integrale mit Hilfe partieller Integration:

$$\text{a)} \int_0^\pi \sin^4 x \, dx \,,$$

$$\text{b)} \int_{-\pi}^\pi \sin^5 x \, dx \,,$$

LÖSUNG:

a) $\int\limits_0^\pi \sin^4 x\,dx = \frac{3\pi}{8}$. Denn: Partielle Integration mit

$$f(x) = -\cos x,\ f'(x) = \sin x,\ g(x) = \sin^3 x,\ g'(x) = 3\sin^2 x \cos x$$

ergibt:

$$\begin{aligned}
\int_0^\pi \sin x \sin^3 x\,dx &= -\cos x \sin^3 x|_0^\pi + 3\int_0^\pi \cos^2 x \sin^2 x\,dx \\
&= 3\int_0^\pi \cos^2 x \sin^2 x\,dx \qquad (\text{da } \sin 0 = \sin x = 0) \\
&= 3\int_0^\pi (1-\sin^2 x)\sin^2 x\,dx \qquad (\cos^2 x + \sin^2 x = 1\,!) \\
&= 3\int_0^\pi \sin^2 x\,dx - 3\int_0^\pi \sin^4 x\,dx. \\
\Rightarrow \int_0^\pi \sin^4 x\,dx &= \frac{3}{4}\int_0^\pi \sin^2 x\,dx. \\
\int_0^\pi \sin^2 x\,dx &= \int_0^\pi \sin x \sin x\,dx \\
&= -\cos x \sin x|_0^\pi + \int_0^\pi \cos^2 x\,dx \\
&= \int_0^\pi \cos^2 x\,dx \quad (-\cos x \sin x|_0^\pi = 0,\ \text{da}\ \sin 0 = \sin\pi = 0) \\
&= \int_0^\pi 1\,dx - \int_0^\pi \sin^2 x\,dx \\
&= \pi - \int_0^\pi \sin^2 x\,dx. \\
\Rightarrow \int_0^\pi \sin^2 x\,dx &= \frac{\pi}{2}, \quad \text{und} \quad \int_0^\pi \sin^4 x\,dx = \frac{3}{4}\cdot\frac{\pi}{2} = \frac{3\pi}{8}.
\end{aligned}$$

b) $\int\limits_{-\pi}^\pi \sin^5 x\ dx = 0$, da $f(x) = \sin^5 x$ ungerade ist: $f(-x) = [\sin(-x)]^5 = [-\sin x]^5 = -\sin^5 x = -f(x)$.