Numerical Simulation

Summer Semester 2015
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Excercise sheet 4.

Closing date **12.05.2015**.

Theoretical exercise 1. (Dual Cones [6 points])

Let U be a Banach space and $K \subseteq U$ a nonempty cone. Recall that the dual cone is defined by

$$K^+ := \{g \in U^* \mid g(u) \geq 0 \; \forall\, u \in K\}.$$

Furthermore for a fixed $f \in U^*$, $f \neq 0$, let us define the cones

$$K_= := \{u \in U \mid f(u) = 0\} \quad \text{and} \quad K_\leq := \{u \in U \mid f(u) \leq 0\}.$$

a) Prove that $(K_=)^+ = \{\lambda f \mid \lambda \in \mathbb{R}\}$ and $(K_\leq)^+ = \{\lambda f \mid \lambda \in (-\infty, 0]\}$.

b) Let K be closed and let $u \in U$ be such that

$$g(u) \geq 0 \;\; \forall\, g \in K^+.$$

Prove that $u \in K$.

Theoretical exercise 2. (Lagrange Duality [6 points])

Let U be a linear space and let Z be a normed space. Let $f : C \to \mathbb{R}$ be a convex function where $C \subseteq U$ is convex. Let furthermore $G : U \to Z$ be convex and let $K \subseteq Z$ be a nonnegative cone. Assume that the Slater condition holds and that

$$\mu_0 := \inf_{x \in C, G(x) \leq_K 0} f(x)$$

is finite.
Let us define $\phi : K^+ \to \mathbb{R}$ by

$$\phi(z^*) := \inf_{x \in C} \left(f(x) + z^*(G(x)) \right).$$

Recall the definition of $\omega : \Gamma \to \mathbb{R} \cup \{-\infty\}$ from the lecture:

$$\omega(z) := \inf\{f(u) \mid u \in C, G(u) \leq_K z\}.$$

Here $\Gamma = \{z \in Z \mid \exists\, u \in C \text{ s.t. } G(u) \leq_K z\}$.

a) Prove that

$$\phi(z^*) = \inf_{z \in \Gamma} (\omega(z) + z^*(z)).$$

b) Prove that

$$\mu_0 = \sup_{z^* \in K^+} \phi(z^*)$$

and that the supremum on the right hand side is attained.

Programming exercise 1. (Newton-Lagrange Method)

The programming exercises have to be done in C/C++. Please mail your code to *bohn@ins.uni-bonn.de* by 12th of May.

Recall theoretical exercise 2 from sheet 1, i.e. consider the constrained minimization problem

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) := 3x^2 + y^2 \quad \text{such that} \quad g(x,y) := \frac{3}{2}x^2 + y = 2.$$

Implement a Newton-Lagrange minimization algorithm for this problem in the following way:

- Implement several functions which return the function values, the first derivatives and the second derivatives of f and g at a given point (x, y).
- Implement Newton's root finding algorithm to detect the zeroes of the derivative of the Lagrangian $\nabla\mathcal{L}$ where $\mathcal{L}(x, y, \lambda) := f(x, y) + \lambda(g(x, y) - 2)$ since these are potential minimizers. The algorithm should stop when the right hand side in the Newton iteration has an ℓ_2 norm smaller than 10^{-15}. You should implement this algorithm on your own and not take a ready-to-use algorithm from a numerical library.
- For the solution of the upcoming linear equation system you may use a solver from a numerical standard library (e.g. *gsl_linalg_LU_solve* from the gsl [`http://www.gnu.org/software/gsl/`]) or your own linear equation system solver.
- Use starting values $x = 5000, y = -3000$ and $\lambda = 400$. What point does the algorithm converge to? How many iterations does it need? Plot the ℓ_2 norm of the right hand side vector in the Newton algorithm as a function of the current iteration number in a semilogarithmic plot (e.g. with *gnuplot*).