Numerical Simulation

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Excercise sheet 8.

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Theoretical exercise 1. (Regularity of optimal control [6 points])

Let Ω be a bounded Lipschitz domain and let $y_\Omega \in L_2(\Omega)$, $u_a, u_b \in H^1(\Omega)$ with $u_a \leq u_b$ almost everywhere and let $\beta \in C^{0,1}(\bar{\Omega})$. Consider the optimal control problem

$$\min_{y \in H^1(\Omega), u \in L_2(\Omega)} \frac{1}{2}\|y - y_\Omega\|^2_{L_2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L_2(\Omega)}$$

(with $\lambda > 0$) such that

$$\begin{aligned} -\Delta y &= \beta u && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma, \\ u_a \leq u &\leq u_b && \text{a.e. on } \Gamma. \end{aligned}$$

Prove that for the optimal control $\bar{u}$ it holds $\bar{u} \in H^1(\Omega)$.

Theoretical exercise 2. (Box constraints and higher order projections [6 points])

For a bounded Lipschitz domain Ω let τ be a quasi-uniform triangulation. Let $U_{\text{ad}} := \{u \in L_2(\Omega) \mid u_a \leq u \leq u_b \text{ almost everywhere}\}$ and let $U_{\text{ad},h} := \{u_h \in U_h \mid u_a \leq u_h \leq u_b\}$ for $u_a, u_b \in L_2(\Omega)$ and a discretized space $U_h \subset L_2(\Omega)$. In constrast to the lecture let now U_h be the space of piecewise linear functions on τ. Prove that

$$u \in U_{\text{ad}} \not\Rightarrow \Pi_h u \in U_{\text{ad},h}$$

by giving a counter-example ($\Omega \subset \mathbb{R}$ suffices). Here, $\Pi_h : L_2(\Omega) \to U_h$ is the L_2-orthogonal projection on U_h.

Theoretical exercise 3. (Projected gradient method for Poisson problems [5 points])

In the setting of Exercise 1 with $\beta \equiv 1$ determine an as large as possible constant $\sigma_{\max} > 0$ - in terms of the norms of the operators S and S^* corresponding to the state equation and the adjoint equation - such that the projected gradient method converges for $\sigma \in (0, \sigma_{\max})$, i.e. calculate λ and L from Theorem 2.28.