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Gauß theorem for tensor divergence

Def. 1 Let $f, g : \Omega \to \mathbb{R}$ be functions,
$u, v : \Omega \to \mathbb{R}^d$ vector fields,
$A : \Omega \to \mathbb{R}^{d \times d}$ a tensor field.

Def. 2 The tensor divergence is defined as

$$(\nabla \cdot A)_i = \sum_j \partial_j A_{ij} \,. \qquad (1)$$

Thm. 3 (Gauß theorem)

$$\forall i = 1, \dots, d: \quad \int_\Omega \partial_i f = \int_{\partial\Omega} \nu_i f \qquad (2)$$

with outside normal vector ν on $\partial\Omega$.

If we set $f = u_i$ and sum over i in (2), we obtain

$$\int_\Omega \nabla \cdot u = \int_{\partial\Omega} \nu \cdot u \,. \qquad (3)$$

Inserting $u = f \nabla g$ into (3) yields

$$\int_\Omega \nabla f \cdot \nabla g = - \int_\Omega f \Delta g + \int_{\partial\Omega} \nu \cdot f \nabla g \,. \qquad (4)$$

This equation is fundamental to the finite element discretitation of Poisson's equation. To discretize Stokes or linear elasticity we have to go one step further.

Note that

$$\nabla\cdot(Av) = \sum_i \partial_i \sum_j A_{ij} v_j =$$

$$= \sum_{ij} \left(v_j \partial_i A_{ij} + A_{ij} \partial_i v_j \right) =$$

$$= v \cdot \nabla\cdot A^T + A:(\nabla v)^T. \qquad (5)$$

Thus using $u = A^T v$ in (3) yields

$$\int_\Omega A:\nabla v = -\int_\Omega v\cdot\nabla\cdot A + \int_{\partial\Omega} v\cdot Av. \quad (6)$$

We have <u>not</u> assumed symmetry of A. The symmetric form of Stokes etc. follows from setting

$$A = \tfrac{1}{2}\left(\nabla u + (\nabla u)^T\right) \implies \qquad (7)$$

$$\int_\Omega A:\nabla v = \int_\Omega \tfrac{1}{2}\left(\nabla u + (\nabla u)^T\right) : \tfrac{1}{2}\left(\nabla v + (\nabla v)^T\right).$$