Wissenschaftliches Rechnen II
(Scientific Computing II)

Sommersemester 2015
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Exercise 38. *(local matrices of the MINI finite element)*
Denote by $\lambda_1, \lambda_2, \lambda_3$ the three nodal P_1 basis functions on a triangle T and define the cubic bubble function $b_T := \lambda_1\lambda_2\lambda_3$. Define the local basis functions of the velocity part of the MINI finite element by

$$\psi_1 = \begin{pmatrix}\lambda_1\\0\end{pmatrix}, \psi_2 = \begin{pmatrix}\lambda_2\\0\end{pmatrix}, \psi_3 = \begin{pmatrix}\lambda_3\\0\end{pmatrix}, \psi_4 = \begin{pmatrix}0\\\lambda_1\end{pmatrix}, \psi_5 = \begin{pmatrix}0\\\lambda_2\end{pmatrix}, \psi_6 = \begin{pmatrix}0\\\lambda_3\end{pmatrix}, \psi_7 = \begin{pmatrix}b_T\\0\end{pmatrix}, \psi_8 = \begin{pmatrix}0\\b_T\end{pmatrix}.$$

The local basis functions for the pressure component are $\lambda_1, \lambda_2, \lambda_3$. The local matrices then read as

$$A_T = \left[\int_T D\psi_j : D\psi_k\,dx\right]_{j,k=1,\dots,8} \quad\text{and}\quad B_T = \left[-\int_T \lambda_j \operatorname{div}\psi_k\,dx\right]_{\substack{j=1,\dots,3\\k=1,\dots,8}}.$$

(a) Prove that A_T has the following block structure

$$A_T = \begin{bmatrix} S & 0 & 0\\ 0 & S & 0\\ 0 & 0 & R\end{bmatrix}$$

for

$$S = \left[\int_T \nabla\lambda_j\cdot\nabla\lambda_k\,dx\right]_{j,k=1,2,3} \quad\text{and}\quad R = \frac{\operatorname{meas}(T)}{180}\sum_{j=1}^{3}|\nabla\lambda_j|^2\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

(b) Prove that B_T has the following block structure

$$B_T = \operatorname{meas}(T)\left[\begin{array}{c|c} L & \\ L & G\\ L & \end{array}\right]$$

for

$$L = -\frac{1}{3}\begin{pmatrix}\partial_x\lambda_1 & \partial_x\lambda_2 & \partial_x\lambda_3 & \partial_y\lambda_1 & \partial_y\lambda_2 & \partial_y\lambda_3\end{pmatrix} \quad\text{and}\quad G = \frac{1}{60}\begin{bmatrix}\partial_x\lambda_1 & \partial_y\lambda_1\\ \partial_x\lambda_2 & \partial_y\lambda_2\\ \partial_x\lambda_3 & \partial_y\lambda_3\end{bmatrix}.$$

Exercise 39. *(global system matrix of the MINI finite element)*
Let $N_N = \operatorname{card}(\mathcal{N}(\Omega))$ denote the number of interior vertices and $N_T = \operatorname{card}(\mathcal{T})$ denote the number of triangles in the triangulation $\mathcal{T}$ of the 2D domain Ω. With the nodal basis functions $(\lambda_z)_{z\in\mathcal{N}}$ define the following basis functions $\psi_1, \dots, \psi_{2N_N+2N_T}$ for the velocity by

$$(\psi_1,\dots,\psi_{N_N}) = \left[\begin{pmatrix}\lambda_z\\0\end{pmatrix}\right]_{z\in\mathcal{N}(\Omega)}, \quad (\psi_{N_N+1},\dots,\psi_{2N_N}) = \left[\begin{pmatrix}0\\\lambda_z\end{pmatrix}\right]_{z\in\mathcal{N}(\Omega)},$$

$$(\psi_{2N_N+1},\dots,\psi_{2N_N+N_T}) = \left[\begin{pmatrix}b_T\\0\end{pmatrix}\right]_{T\in\mathcal{T}}, \quad (\psi_{2N_N+N_T+1},\dots,\psi_{2N_N+2N_T}) = \left[\begin{pmatrix}0\\b_T\end{pmatrix}\right]_{T\in\mathcal{T}}.$$

(for the definition of b_T see Exercise 38) and for the pressure component define

$$(q_1,\dots,q_{N_N}) = [\lambda_z]_{z\in\mathcal{N}(\Omega)}.$$

The global matrices then read as

$$A = \left[\int_\Omega D\psi_j : D\psi_k \, dx \right]_{j,k=1,\ldots,2(N_N+N_T)} \quad \text{and} \quad B = \left[-\int_\Omega q_j \operatorname{div} \psi_k \, dx \right]_{\substack{j=1,\ldots,N_N \\ k=1,\ldots,2(N_N+N_T)}}.$$

(a) Prove that the global system matrix M of the MINI element has the block structure

$$M = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

so that the discrete equation reads as

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

(b) Prove that the matrix M has a nontrivial kernel.

Hint: Consider globally constant pressure modes.

(c) Include the constraint $\int_\Omega p_h \, dx = 0$ in the global system by using a Lagrange multiplier.

Exercise 40. *(backward facing step)*
Download the Mini element software from the course webpage. An example is given in the file `colliding_flow.m`. Use the Mini element to simulate the flow over a backward facing step. Print the computed velocity and pressure and present the plots in the tutorial session. The parameters are:

- Domain: $\Omega = ((-2, 8) \times (-1, 1)) \setminus ([-2, 0] \times [-1, 0])$ (see Figure 1)
- Forcing term: $f = 0$,
- Dirichlet data: $u_D(x, y) = \begin{cases} (0,0) & \text{for } -2 < x < 8 \\ (-y(y-1)/10, 0) & \text{for } x = -2 \\ (-(y+1)(y-1)/80, 0) & \text{for } x = 8. \end{cases}$

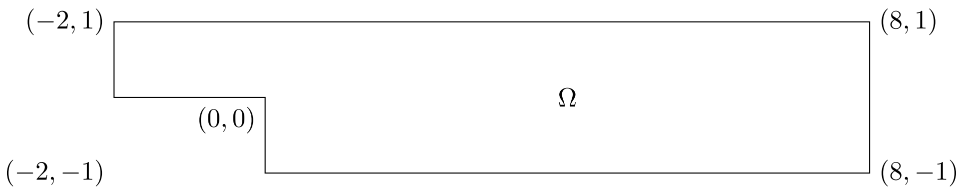

Figure 1: The backward facing step.

Exercise 41. *(conforming companion operators)*
The design of the conforming companions to any $v_h \in \mathrm{CR}_0^1$ begins with the map $J_1 : \mathrm{CR}_0^1(\mathcal{T}) \to P_0^{1,0}(\mathcal{T})$ defined by

$$J_1 v_h := \sum_{z \in \mathcal{N}(\Omega)} \operatorname{card}(\mathcal{T}(z))^{-1} \sum_{T \in \mathcal{T}(z)} v_h|_T(z) \, \lambda_z,$$

where λ_z denotes the conforming nodal basis function. For a given interior edge $E := \operatorname{conv}\{a, b\} \in \mathcal{E}(\Omega)$ let $b_E := 6\lambda_a \lambda_b$ denote the edge bubble function. Then the operator $J_2 : \mathrm{CR}_0^1((T)) \to P_0^{2,0}(\mathcal{T})$ is given by

$$J_2 v_h := J_1 v_h + \sum_{E \in \mathcal{E}(\Omega)} \left(\fint_E (v_h - J_1 v_h) \, ds \right) b_E.$$

For any triangle $T \in \mathcal{T}$ with $T = \operatorname{conv}\{a, b, c\}$ define the element bubble function $b_T := 60\lambda_a\lambda_b\lambda_c$. The operator $J_3 : \mathrm{CR}_0^1((T)) \to P_0^{3,0}(\mathcal{T})$ is given by

$$J_3 v_h := J_2 v_h + \sum_{T\in\mathcal{T}} \left(⨍_T (v_h - J_2 v_h)\, dx \right) b_T.$$

Prove that the operators $J_k : \mathrm{CR}_0^1(\mathcal{T}) \to P_0^{k,0}(\mathcal{T})$, $k = 1, 2, 3$, defined above satisfy

(a) the conservation properties

$$\int_T \nabla(v_h - J_k v_h)\, dx = 0 \qquad \text{for all } T \in \mathcal{T} \text{ and } k = 2, 3,$$

$$\int_T (v_h - J_3 v_h)\, dx = 0 \qquad \text{for all } T \in \mathcal{T},$$

(b) the approximation and stability properties for $k = 1, 2, 3$

$$\|h_{\mathcal{T}}^{-1}(v_h - J_k v_h)\| \approx \|\nabla_{\mathrm{NC}}(v_h - J_k v_h)\| \approx \min_{\varphi\in H_0^1(\Omega)} \|\nabla_{\mathrm{NC}}(v_h - \varphi)\|.$$