Wissenschaftliches Rechnen II
(Scientific Computing II)

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Exercise 42. *(condition numbers)*
Let $\mathcal{T}$ be a quasi-uniform triangulation of the domain $\Omega \subseteq \mathbb{R}^d$, that is, there are constants $0 < c \leq C < \infty$ such that

$$c \leq \frac{\max_{T\in\mathcal{T}} \operatorname{diam}(T)}{\min_{T\in\mathcal{T}} \operatorname{diam}(T)} \leq C.$$

Let $h = \max_{T\in\mathcal{T}} \operatorname{diam}(T)$. Let $(\lambda_z)_{z\in\mathcal{N}(\Omega)}$ denote the nodal basis of $P_0^{1,0}(\mathcal{T})$ and recall the mass and stiffness matrices

$$M = \left[\int_\Omega \lambda_y \lambda_z \, dx\right]_{y,z\in\mathcal{N}(\Omega)} \quad \text{and} \quad S = \left[\int_\Omega \nabla\lambda_y \cdot \nabla\lambda_z \, dx\right]_{y,z\in\mathcal{N}(\Omega)}.$$

(a) Denote by $\mu_{\min}$ and $\mu_{\max}$ the smallest and the largest eigenvalue of M. Prove that there are mesh-size independent, positive constants ρ_1, ρ_2 such that $\rho_1 h^d \leq \mu_{\min} \leq \mu_{\max} \leq \rho_2 h^d$.

(b) Prove that the spectral condition number $\kappa_2(M) := \sqrt{\mu_{\max}/\mu_{\min}}$ is independent of h.

(c) Denote by $\sigma_{\min}$ and $\sigma_{\max}$ the smallest and and the largest eigenvalue of S. Prove that there are mesh-size independent, positive constants η_1, η_2 such that $\eta_1 h^d \leq \sigma_{\min} \leq \sigma_{\max} \leq \eta_2 h^{d-2}$. Prove that these estimates are sharp.

(d) Prove that the spectral condition number $\kappa_2(S) := \sqrt{\sigma_{\max}/\sigma_{\min}}$ can be bounded from above by $\mathcal{O}(h^{-2})$. Prove that this estimate is sharp.

Hint: 1. Recall the Rayleigh quotient characterization of minimal/maximal eigenvalues. 2. Use the Friedrichs inequality and inverse estimates in (c).

Exercise 43. *(Leapfrog for the 2D wave equation)*
Write a program that numerically solves the wave equation

$$u_{tt} = \Delta u \text{ in } \Omega \times (0,T), \quad u(\cdot,t) = 0 \text{ on } \partial\Omega, \quad u(0,\cdot) = u_0 \text{ on } \partial\Omega, \quad u_t(0,\cdot) = u_0 \text{ on } \partial\Omega$$

for $\Omega \subseteq \mathbb{R}^2$. Use the P_1 finite element method for the spacial discretization and the Leapfrog method for the time integration.
Hint: The system matrices (mass, stiffness) for the space discretization can be downloaded on the course website (e.g. Exercise Sheet 7).

Exercise 44. *(simulation of 2D wave equation)*
Use the software from Exercise 43 to solve the wave equation on $\Omega = (0,1)^2$ with the following initial data

(a)
- $u_0(x_1,x_2) = \sin(2\pi x_1)\sin(2\pi x_2)$
- $u_1(x_1,x_2) = 0$

(b)
- $u_0(x_1,x_2) = x_1 \sin(8\pi x_1)\sin(3\pi x_2)$
- $u_1(x_1,x_2) = \exp(1 + x_2^2)$

Produce movies of you computed solution by using the program `getMovie4Wave(T,U,speed,filename)` (download from the course website), where `T` is the triangulation and the matrix `U` contains the solution for each time step in its columns. (The parameters `speed` and `filename` are optional.)

Exercise 45. *(mass lumping)*
Let $\mathcal{T}$ denote a triangulation of $\Omega \subseteq \mathbb{R}^2$ with nodal basis function $(\lambda_z)_{z\in\mathcal{N}}$. Denote by I the nodal P_1 interpolation operator. Define the matrix

$$L = \left[\int_\Omega I_h(\lambda_y \lambda_z)\, dx\right]_{y,z\in\mathcal{N}}.$$

(a) Prove that L is diagonal.

(b) Recall the mass matrix $M = \left[\int_\Omega \lambda_y \lambda_z\, dx\right]_{y,z\in\mathcal{N}}$ and prove that

$$L_{jj} = \sum_{k=1}^{\operatorname{card}(\mathcal{N})} M_{jk},$$

that is the diagonal entries of L are the sums of the rows of M.

Remark: This is why L is referred to as "lumped" mass matrix.

(c) Run the numerical experiments of Exercise 44 with the lumped version of the mass matrix and compare the results.