# Wissenschaftliches Rechnen II
# (Scientific Computing II)

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**Exercise 6.** *(Banach-Babuška-Nečas theorem)*
Let $V$ be a Banach space and let $W$ be a reflexive Banach space and let $a \in \mathcal{L}(V \times W)$ satisfy

$$0 < \alpha = \inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{a(v,w)}{\|v\|_V \|w\|_W}, \tag{BBN1}$$

$$\forall w \in W \setminus \{0\}\, \exists v \in V \; a(v,w) \neq 0. \tag{BBN2}$$

Define furthermore $A_1 \in \mathcal{L}(V, W')$ and $A_2 \in \mathcal{L}(W, V')$ by $A_1(v) := a(v, \cdot)$ and $A_2(w) := a(\cdot, w)$.

(a) Prove that the range of $A_1$ is closed in $W'$.

(b) Prove that the range of $A_1$ equals $W'$.

(c) Prove that $A_1$ is an isomorphism and $\|A_1^{-1} F\|_V \leq \alpha^{-1} \|F\|_{W'}$ for any $F \in W'$.

(d) Prove that $A_2$ is an isomorphism with $\|A_1^{-1}\|_{\mathcal{L}(W';V)} = \alpha^{-1} = \|A_2^{-1}\|_{\mathcal{L}(V';W)}$.

(e) Prove that

$$\inf_{w \in W \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{a(v,w)}{\|v\|_V \|w\|_W} = \alpha.$$

**Exercise 7.** *(energy functional)*
Let $H$ be a Hilbert space and $a : H \times H \to \mathbb{R}$ a symmetric bilinear form that induces the norm $\| \cdot \|_a^2 = a(\cdot, \cdot)$ on $H$. Given $b \in H'$, define the quadratic functional

$$E(v) = \frac{1}{2} a(v,v) - b(v) \quad \text{for } v \in H.$$

Prove that $u \in H$ satisfies $a(u,v) = b(v)$ for all $v \in H$ if and only if $u \in H$ is the unique minimizer of $E$ over $H$.

**Exercise 8.** *(error computation)*
Given $F \in H^{-1}(\Omega)$, define the energy functional

$$E(v) := \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 - F(v) \quad \text{for } v \in H_0^1(\Omega).$$

Prove that the error of the conforming finite element method for the Poisson problem with right-hand side $F$ satisfies

$$\|\nabla(u - u_h)\|_{L^2(\Omega)}^2 = 2(E(u_h) - E(u)) = \|\nabla u\|_{L^2(\Omega)}^2 - \|\nabla u_h\|_{L^2(\Omega)}^2.$$

**Exercise 9.** *(element matrices)*
Let $T$ be a triangle with with set of vertices $\mathcal{N}(T)$. Given $y \in \mathcal{N}(T)$, denote by $\lambda_y \in P_1(T)$ the affine function defined by

$$\lambda_y(z) = \delta_{yz} \quad \text{for all } z \in \mathcal{N}(T).$$

Compute the following $3 \times 3$ matrices

$$M_T := \left( \int_T \lambda_y \lambda_z \, dx \right)_{(y,z) \in (\mathcal{N}(T))^2} \qquad \text{(local mass matrix)}$$

$$C_T := \left( \int_T \lambda_y (\beta \cdot \nabla \lambda_z) \, dx \right)_{(y,z) \in (\mathcal{N}(T))^2} \qquad \text{(local convection matrix with given } \beta \in \mathbb{R}^2\text{)}$$

$$S_T := \left( \int_T \nabla \lambda_y \cdot \nabla \lambda_z \, dx \right)_{(y,z) \in (\mathcal{N}(T))^2} \qquad \text{(local stiffness matrix)}$$

on the reference triangle $T := \operatorname{conv}\{(0,0),(1,0),(0,1)\}$.