Sheet 5 due date: **11. Mai 2015**

Exercise 18. *(Robin boundary conditions)*
Let $\mathcal{T}$ be a regular triangulation of the open bounded domain $\Omega \subseteq \mathbb{R}^2$. The Poisson equation with Robin boundary conditions

$$-\Delta u = f \quad \text{in } \Omega \qquad u + \frac{\partial u}{\partial \nu} = 0 \tag{1}$$

is stated in its week form as: Seek $u \in H^1(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\partial\Omega)} = (f, v)_{L^2(\Omega)} \qquad \text{for all } v \in H^1(\Omega). \tag{2}$$

(a) Prove that the left-hand side of (2) is coercive on $H^1(\Omega)$ and that (2) is well-posed. Prove furthermore that solutions to (2) solve (1) in a weak sense.

(b) Given the real interval $[0, h]$ for $h > 0$ and the affine functions λ_1^{1D}, λ_2^{1D} defined by the nodal values

$$\lambda_1^{1D}(0) = \lambda_2^{1D}(h) = 1 \quad \text{and} \quad \lambda_1^{1D}(h) = \lambda_2^{1D}(0) = 0$$

i.e., the nodal basis of $P^1([0, h])$, compute the (local) 1D mass matrix M_{1D} defined by

$$M_{1D} = \left(\int_0^h \lambda_j^{1D}(x) \lambda_k^{1D}(x)\, dx \right)_{j,k=1,2}.$$

(c) Write a Matlab or Octave program that assembles the boundary mass matrix B defined by

$$B_{jk} = \int_{\partial\Omega} \lambda_j \lambda_k \, ds$$

for the nodal basis $(\lambda_j)_j$ of $P^{1,0}(\mathcal{T})$.

Hint: Note that most entries of B are zero. Implement a loop over the boundary edges.

(d) Implement the finite element method for (2) in Matlab or Octave by modifying the code of Exercise Sheet 3.

(e) Compute finite element approximations to (2) for $f \equiv 1$. Print a surface plot of the computed solution.

Exercise 19. *(continuity of conforming FEM functions)*
Let $\mathcal{T}$ be a regular triangulation of $\Omega \subseteq \mathbb{R}^d$ und let $v \in P^{k,-1}(\mathcal{T})$ (for $k \in \mathbb{N}$) be a piecewise polynomial function. For each interior $(d-1)$ dimensional hyper-face F (edges for $d = 2$ or faces for $d = 3$) with adjacent simplices T_+ and T_- (i.e., $F = T_+ \cap T_-$), the jump across F is defined by $[v]_F := v|_{T_+} - v|_{T_-}$.

(a) Prove that

$$v \in H^1(\Omega) \iff [v]_F = 0 \quad \text{for all interior hyper-faces } F$$

(b) The space $H(\text{div}, \Omega)$ is defined by

$$H(\text{div}, \Omega) := \left\{ v \in L^2(\Omega; \mathbb{R}^2) \;\middle|\; \begin{array}{l} \exists g \in L^2(\Omega) \text{ such that for all } \varphi \in \mathcal{D}(\Omega) \\ \int_\Omega v \cdot \nabla\varphi \, dx = -\int_\Omega g\varphi \, dx \end{array} \right\}.$$

Prove that

$$v \in H(\text{div}, \Omega) \iff [v \cdot \nu_F]_F = 0 \quad \text{for all interior hyper-faces } F$$

where ν_F is some normal vector of F.

Exercise 20. *(smallest singular value = inf-sup constant)*
Let $X := \mathbb{R}^m$ and $Y := \mathbb{R}^n$ be finite dimensional and $b : X \times Y \to \mathbb{R}$ be a bilinear form. Given orthonormal bases $(\xi_1, \dots, \xi_m)$ and $(\eta_1, \dots, \eta_n)$ of X and Y, respectively, the matrix representation of b reads as $B = (B_{jk})_{\substack{j=1,\dots,m \\ k=1,\dots,n}} \in \mathbb{R}^{m \times n}$ with

$$B_{jk} := b(\xi_j, \eta_k) \quad \text{for all } (j,k) \in \{1, \dots, m\} \times \{1, \dots, n\}.$$

Prove that the inf-sup constant

$$\beta := \inf_{x \in X \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{b(x,y)}{\|x\| \, \|y\|}$$

equals the smallest singular value of B.
Reminder: For any real $m \times n$ matrix M there exist matrices $U \in O(m)$ und $V \in O(n)$ and a diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ with non-negative entries such that $M = U \Sigma V$. The positive entries of Σ are called the singular values of M.