Wissenschaftliches Rechnen II
(Scientific Computing II)

Sommersemester 2015
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Exercise 21. *(FEM in 1D)*

(a) Prove that in one space dimension the approximation of the equation $u''(x) = 1$ (on the interval $(0,1)$ with homogeneous Dirichlet boundary conditions) with the P_1 finite element method results in the nodal interpolation, that is $u_h = I_h u$.

(b) Convince yourself that this property cannot be valid in higher space dimensions (e.g., by comparing with Exercise 4).

Exercise 22. *(oscillatory coefficient)*
Consider the unit interval $\Omega = (0,1)$ and the second-order boundary value problem

$$-\frac{d}{dx}\left(A_\varepsilon(x)\frac{d}{dx}u_\varepsilon(x)\right) = 1 \quad \text{in } \Omega \qquad \text{and} \qquad u_\varepsilon(0) = u_\varepsilon(1) = 0 \tag{1}$$

for the coefficient

$$A_\varepsilon(x) := \left(2 + \cos\left(2\pi\frac{x}{\varepsilon}\right)\right)^{-1}.$$

The weak formulation for this specific right-hand side seeks $u_\varepsilon \in H^1_0(\Omega)$ such that

$$\int_0^1 A_\varepsilon(x) u_\varepsilon'(x) v'(x)\,dx = \int_0^1 v(x)\,dx \quad \text{for all } v \in H^1_0(\Omega).$$

(a) Verify that the solution to (1) is given by

$$u_\varepsilon(x) = x - x^2 - \varepsilon\left(\frac{1}{4\pi}\sin\left(2\pi\frac{x}{\varepsilon}\right) - \frac{x}{2\pi}\sin\left(2\pi\frac{x}{\varepsilon}\right) - \frac{\varepsilon}{4\pi^2}\cos\left(2\pi\frac{x}{\varepsilon}\right) + \frac{\varepsilon}{4\pi^2}\right).$$

(b) Let $N \in \mathbb{N}$ and $\varepsilon = 1/N$. Consider the finite element mesh

$$\mathcal{T} = \{[kh, (k+1)h] : k = 0, 1, \dots N\}$$

with mesh-size $h = \varepsilon$. The finite element method seeks $u_h \in V_h$ such that

$$\int_0^1 A_\varepsilon(x) u_{\varepsilon,h}'(x) v_h'(x)\,dx = \int_0^1 v_h(x)\,dx \quad \text{for all } v_h \in V_h.$$

Prove that $u_{\varepsilon,h}$ coincides with the nodal interpolation of the function

$$w(x) = -\frac{\sqrt{3}}{2}(x^2 - x).$$

Hint: Use Exercise 21. You may also use that $\int_0^\varepsilon A_\varepsilon(x)\,dx = \varepsilon/\sqrt{3}$.

(c) Plot (with Matlab/Octave or by hand) the functions u_ε, $u_{\varepsilon,h}$ as well as the nodal interpolation of u_ε for $h = \varepsilon = 0.1, 0.01$. Does $u_{\varepsilon,h}$ provide a meaningful approximation to u_ε in an L^2 sense? What happens for coarser mesh sizes $h = k\varepsilon$ ($k \in \mathbb{N}$)?

(d) Give a (crude) error estimate for small values of ε that proves that the error in the L^2 norm $\|u_\varepsilon - u_{\varepsilon,h}\|_{L^2(\Omega)}$ is not better than order $\mathcal{O}(1)$ when $h = \varepsilon$.

(e) Compute a guaranteed upper bound for the error $\|u_\varepsilon - u_{\varepsilon,h}\|_{H^1(\Omega)}$ (e.g., by using Céa's lemma). What can you conclude for small values of $h \ll \varepsilon$?

Exercise 23. *(jumping coefficient I)*
Consider the unit interval $\Omega = (0,1)$ and the second-order boundary value problem

$$-\frac{d}{dx}\left(A(x)\frac{d}{dx}u(x)\right) = 1 \quad \text{in } \Omega \qquad \text{and} \qquad u(0) = u(1) = 0$$

for the coefficient (with $\gamma > 1$)

$$A(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq 0.5, \\ \gamma & \text{if } 0.5 < x \leq 1. \end{cases}$$

(a) Compute u analytically and plot its graph (for $\gamma = 10$).

(b) Prove that u and Au' are continuous, but u' is discontinuous. Discuss why the "kink" in the solution is aligned with the discontinuity of A.

Exercise 24. *(jumping coefficient II)*
Let $\Omega \subseteq \mathbb{R}^2$ and let $A \in L^\infty(\Omega)$ with uniform bounds $\underline{\alpha}$, $\overline{\alpha}$ such that

$$0 < \underline{\alpha} \leq A(x) \leq \overline{\alpha} < \infty \quad \text{a.e. in } \Omega.$$

Consider the PDE

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega \qquad u|_{\partial\Omega} = 0.$$

(a) Implement the finite element method for this problem in the case of a piecewise constant $A \in P^{0,-1}(\mathcal{T})$ by modifying the software from Exercise Sheet 3.

(b) Compute the finite element solution for the unit square domain $\Omega = (0,1)^2$ and the coefficient function

$$A(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}] \times [0, \frac{1}{2}], \\ 20 & \text{else.} \end{cases}$$

Plot the computed solution. Can one recognize where the coefficient A is discontinuous?