Wissenschaftliches Rechnen II
(Scientific Computing II)

Sommersemester 2015
Prof. Dr. Daniel Peterseim
Dr. Dietmar Gallistl

Sheet 7 due date: **01. Juni 2015**

01.Juni 2015: Tutorial in the computer lab 6.012
Solutions to be presented during the lab session

Exercise 25. *(Banach-Babuška-Nečas theorem for complex vector spaces)*
Let V, W be Banach spaces over $\mathbb{C}$ with W reflexive. Let $a : V \times W \to \mathbb{C}$ be a continuous sesquilinear form satisfying

$$0 < \alpha = \inf_{v \in V \setminus \{0\}} \sup_{w \in W \setminus \{0\}} \frac{\operatorname{Re} a(v,w)}{\|v\|_V \|w\|_W}, \tag{BBN1}$$

$$\forall w \in W \setminus \{0\} \; \exists v \in V \; a(v,w) \neq 0. \tag{BBN2}$$

Let $\overline{W}'$ denote the set of continuous antilinear functionals over W. The form a defines the linear map $A_1 : V \to \overline{W}'$ and the antilinear map $A_2 : W \to V'$ via $A_1(v) := a(v,\cdot)$ and $A_2(w) := a(\cdot,w)$ for all $v \in V$, $w \in W$.

(a) Prove that A_1 and A_2 are isomorphisms and that

$$\inf_{w \in W \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{\operatorname{Re} a(v,w)}{\|v\|_V \|w\|_W} = \alpha.$$

(b) Given any bounded antilinear functional $F \in \overline{W}'$ there exists a unique $u \in V$ such that $a(u,\cdot) = F$. Furthermore $\|u\|_V \leq \alpha^{-1} \|F\|_{\overline{W}'}$.

Reminder. A map φ between complex vector spaces X, Y is called antilinear if it is additive and satisfies for all $\mu \in \mathbb{C}$ and all $x \in X$ that $\varphi(\mu x) = \bar{\mu}\varphi(x)$.

Exercise 26. *(runtime)*
Run the FEM code from Exercise Sheet 3 for different numbers of degrees of freedom (`ndof`) and measure the runtime of each experiment.

(a) How does the runtime scale in dependence of `ndof`?

(b) Download the new software for this exercise based on the function `local2global.m` and run analogous experiments. Compare the runtimes.

Hint. You may use the commands `tic` and `toc` for the measurement (works in Matlab and Octave). The `profile` command is also helpful (currently only available in Matlab).

Remark. During the tutorials we will discuss the detalis on the function `local2global.m`.

Exercise 27. *(SDFEM)*
Consider the convection-diffusion equation $-\varepsilon\Delta u + \beta \cdot \nabla u = f$. You can download the FEM code for this problem from the website (explanations were given during the discussion of Exercise 11).

The Streamline-Diffusion FEM (SDFEM) is based on some real parameter $\delta > 0$ and utilizes the following modified discrete formulation

$$\varepsilon(\nabla u_h, \nabla v_h)_{L^2(\Omega)} + \delta \sum_{T \in \mathcal{T}_h} \int_T (\nabla u_h \cdot \beta)(\nabla v_h \cdot \beta)\, dx + (\beta \cdot \nabla u_h, v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in V_h,$$

where V_h is the standard first-order finite element space with homogeneous Dirichlet boundary conditions.

(a) Implement the SDFEM by modifying the FEM code for the convection-diffusion problem (download from the website).

(b) Consider the unit square $\Omega = (0,1)^2$ with homogeneous Dirichlet boundary conditions. Define the function $\eta : [0,1] \to \mathbb{R}$ through

$$\eta(x) = x - \frac{\exp(x/\varepsilon) - 1}{\exp(1/\varepsilon) - 1} \quad \text{for all } x \in [0,1].$$

Let $\beta = (1,1)$ and the right-hand side f be defined by $f(x) = \eta(x_1) + \eta(x_2)$ for all $x = (x_1, x_2) \in \Omega$. The exact solution is given by

$$u(x) = \eta(x_1)\eta(x_2).$$

Consider the L^2 error between u_h and the interpolation Iu as an error measure. Values of h corresponding to 5 or 6 uniform refinements ($h = \sqrt{2} \times 2^{-5}$ or $\sqrt{2} \times 2^{-6}$) should run in reasonable time.

(i) Run numerical computations for $\varepsilon = 0.005$ and various values of δ. Which value of δ gives the best results?

(ii) Create convergence history plots (error versus `ndof` in a loglog diagram) for your experiments.

(iii) Perform parameter studies and determine an appropriate coupling of δ and h.

Remark. In Matlab/Octave you can export the graphical output to pdf, eps, etc.