Wissenschaftliches Rechnen II
(Scientific Computing II)

Sommersemester 2015
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Sheet 8 due date: **08. Juni 2015**

08.Juni 2015: Tutorial in the computer lab 6.012
Solutions to be presented during the lab session

Exercise 28. *(Helmholtz equation)*
Let $\Omega \subseteq \mathbb{R}^2$ with $\Gamma_R := \partial\Omega$. Let $V := H^1(\Omega; \mathbb{C})$. Given a parameter $\kappa > 0$ and $g \in L^2(\Gamma; \mathbb{C})$, the Helmholtz equation with impedance boundary conditions seeks $u \in V$ such that

$$\begin{aligned} -\Delta u - \kappa^2 u &= f \quad \text{in } \Omega, \\ i\kappa u - \nabla u \cdot \nu &= g \quad \text{on } \Gamma_R. \end{aligned} \tag{1}$$

Here $i = \sqrt{-1}$. Define on V the following sesquilinear form

$$a(v, w) := (\nabla v, \nabla w)_{L^2(\Omega)} - \kappa^2 (v, w)_{L^2(\Omega)} - i\kappa (v, w)_{L^2(\Gamma_R)}.$$

The weak form of the Helmholtz problem then seeks $u \in V$ such that

$$a(u, v) = (f, v)_{L^2(\Omega)} + (g, v)_{L^2(\Gamma_R)} \quad \text{for all } v \in V. \tag{2}$$

(a) Implement the finite element method for (2) by modifying the software from Exercise Sheet 7.
Hint: See also Exercise 18 for the mass matrix on the boundary.

(b) On the unit square $\Omega = (0, 1)^2$, consider the pure Robin problem $\Gamma_R = \partial\Omega$ with data

$$f = 0 \quad \text{and} \quad g = i\kappa(1 + \left(\begin{smallmatrix} 0.6 \\ 0.8 \end{smallmatrix}\right) \cdot \nu) w$$

given by the plane wave $w(x) = \exp(-i\kappa x \cdot \left(\begin{smallmatrix} 0.6 \\ 0.8 \end{smallmatrix}\right))$. Here ν is the outward pointing unit normal of Ω. Compute finite element approximations of (1) for $\kappa \in \{2^3, 2^4, 2^5, 2^6, 2^7\}$ for different mesh-sizes and generate surface plots of the discrete solutions (real part).

(c) Verify that the exact solution of the problem from (b) is given by $u = w$. Compare the interpolation $I_h u$ with the discrete solutions, e.g., by comparing the surface plots of the real parts. Create convergence history plots of the error quantity

$$\sqrt{\|\nabla(u_h - I_h u)\|^2_{L^2(\Omega)} + \kappa^2 \|u_h - I_h u\|^2_{L^2(\Omega)}}$$

in dependence of the degrees of freedom (`loglog` plot).

(d) Verify that the function $\Phi(x, t) = u(x) \exp(-i\kappa t)$ satisfies the time-periodic wave equation

$$\Delta\Phi(x, t) - \partial_{tt}\Phi(x, t) = -f(x) \exp(-i\kappa t)$$

with the Robin boundary condition in space. Download the file `getMovie4scattering.m`. Given your discrete solution `x` to (1) on the mesh `T`, the call

```
getMovie4scattering(T,x,20,'filename')
```

produces a movie of the reconstructed approximation of the solution the the corresponding wave equation. Produce movie files for all your computed discrete solutions as well as for the interpolated exact solutions and compare them.

Exercise 29. *(scattering)*
Consider the square with hole $\Omega = (0, 1)^2 \setminus ([0.4, 0.6]^2)$. Consider equation (1) with the additional Dirichlet boundary condition $u = 0$ on Γ_D where Γ_D is the inner boundary of the domain, i.e., $\Gamma_D := \partial([0.4, 0.6]^2)$ and Γ_R is the outer boundary, i.e., $\Gamma_R = \partial([0, 1]^2)$ with Robin data g from Exercise 28(b). Implement the finite element method for this problem and produce movies for the solution to the corresponding wave equation for $\kappa \in \{2^3, 2^4, 2^5, 2^6, 2^7\}$.