Wissenschaftliches Rechnen II
(Scientific Computing II)

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Exercise 30. *(Quasi-interpolation)*
Let $\mathcal{T}_h$ be a shape-regular triangulation of $\Omega \subseteq \mathbb{R}^d$. We define the operator $R_h : H_0^1(\Omega) \to P_0^{1,0}(\mathcal{T}_h)$ via

$$R_h v = \sum_{z \in \mathcal{N}_h(\Omega)} \fint_{\omega_z} v \, dx \, \lambda_z .$$

Here, $\mathcal{N}_h(\Omega)$ is the set of interior vertices, $\{\lambda_z\}_{z \in \mathcal{N}(\Omega)}$ is the nodal basis, and $\omega_z := \cup\{K \in \mathcal{T}_h : z \in K\}$ denotes the nodal patch of a vertex $z \in \mathcal{N}_h(\Omega)$. Prove for any $T \in \mathcal{T}_h$ and any $v \in H_0^1(\Omega)$ that

$$\operatorname{diam}(T)^{-1} \|v - R_h v\|_{L^2(T)} + \|\nabla R_h v\|_{L^2(T)} \leq C \|\nabla v\|_{L^2(\omega_T)}$$

where the element patch ω_T of T is $\omega_T := \cup\{K \in \mathcal{T}_h : K \cap T \neq \emptyset\}$.

Exercise 31. *(inf-sup condition)*
Let V and M be Hilbert spaces and let $b : V \times M \to \mathbb{R}$ be a continuous bilinear form satisfying

$$0 < \beta \leq \inf_{\mu \in M \setminus \{0\}} \sup_{v \in V \setminus \{0\}} b(v, \mu).$$

Define the set

$$Z = \{v \in V : b(v, \mu) = 0 \text{ for all } \mu \in M\}$$

with V-orthogonal complement $Z^\perp$ and polar set $Z^\circ = \{F \in V' : F|_Z = 0\}$. Define the operators $B : Z^\perp \to M'$ and $B' : M \to Z^\circ$ via

$$B(v) = b(v, \cdot) \qquad B'(\mu) = b(\cdot, \mu) \qquad \text{for any } v \in Z^\perp \text{ and } \mu \in M.$$

(a) Check that B and B' are well-defined.

(b) Prove that B is an isomorphism and $\|B^{-1}\|_{\mathcal{L}(M', Z^\perp)} \leq \beta^{-1}$

(c) Prove that B' is an isomorphism and $\|(B')^{-1}\|_{\mathcal{L}(Z^\circ, M)} \leq \beta^{-1}$

Hint: Closed range theorem.

Exercise 32. *(Brezzi splitting is necessary)*
Let V and M be Hilbert spaces with continuous bilinear forms $a : V \times V \to \mathbb{R}$ and $b : V \times M \to \mathbb{R}$. Assume that for any $F \in V'$ and $G \in M'$ there exists a unique solution (u, λ) to the saddle-point problem

$$\left.\begin{array}{ccccc} a(u,v) & + & b(v,\lambda) & = & F(v) \\ b(u,\mu) & & & = & G(\mu) \end{array}\right\} \text{ for all } (v,\mu) \in V \times M$$

and that it satisfies $\|u\|_V + \|\lambda\|_M \leq C(\|F\|_{V'} + \|G\|_{M'})$. Let

$$Z := \{v \in V : b(v, \mu) = 0 \text{ for all } \mu \in M\}.$$

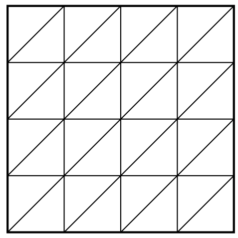
Prove that there exist constants $\alpha, \beta > 0$ such that

$$\inf_{v \in Z \setminus \{0\}} \sup_{w \in Z \setminus \{0\}} \frac{a(v,w)}{\|v\|_V \|w\|_V} \geq \quad \alpha \quad \leq \inf_{v \in Z \setminus \{0\}} \sup_{w \in Z \setminus \{0\}} \frac{a(w,v)}{\|v\|_V \|w\|_V}$$

and

$$\inf_{\mu \in M \setminus \{0\}} \sup_{v \in V \setminus \{0\}} \frac{b(v,\mu)}{\|v\|_V \|\mu\|_M} \geq \beta.$$

Exercise 33. *(Conforming divergence-free functions are trivial)*
Let $\mathcal{T}_h$ be the criss triangulation of the unit square and let $u_h \in P_0^{1,0}(\mathcal{T}_h)$ with $\operatorname{div} u_h = 0$. Prove that $u_h = 0$.
Hint: The criss triangulation is



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