Numerical Simulation

Summer semester 2016
Lecturer: Prof. Dr. Ira Neitzel
Assistant: Dr. Guanglian Li

universitätbonn

Exercise Sheet 1

Closing date **n.a.**.

Exercise 1. (Properties of level sets and existence of solutions)

a) Prove that for a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with $\lim\limits_{\|x\|\to\infty} f(x) = \infty$ and arbitrary $w \in \mathbb{R}^n$ the level set $\mathcal{N}(f, f(w))$ is compact. What does this imply for the solvability of the problem $\min\limits_{x\in\mathbb{R}^n} f(x)$?

b) Show that the level sets of convex functions $f\colon \mathbb{R}^n \to \mathbb{R}$ are convex. Is the opposite also true?

c) Consider the functions

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 - y^2$$

and

$$g\colon \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 + y^2.$$

Sketch the contour lines of f and g. Do f and g reach their maximal and/or minimal value on

$$D := \big\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\big\}?$$

If so, where?

Exercise 2. (Optimality conditions for convex problems)

Consider the convex optimization problem

$$\min_{x\in X} f(x) \qquad (P_{\text{convex}})$$

with a nonempty convex set $X \subset \mathbb{R}^n$ and convex function $f\colon X \to \mathbb{R}$.
Let f be differentiable on an open neighborhood D of the convex set X, and let $\bar{x} \in X$. Prove that $\bar{x}$ is solution of the convex minimization problem (P_{convex}) if and only if the following **variational inequality** is fulfilled,

$$\nabla f(\bar{x})^T(x - \bar{x}) \geq 0 \quad \forall x \in X. \qquad (1)$$

Exercise 3. (Projection on convex sets)

Consider a convex, closed, nonempty set $C \subset \mathbb{R}^n$ and a point $y \in \mathbb{R}^n$. Find a point $x \in C$ - the projection of y onto the set C, - with minimal Euclidean distance from y of all points of C, i.e. a solution of

$$\min_{x\in C} \frac{1}{2}\|x - y\|^2.$$

a) Prove existence and uniqueness of a global solution and state the first order necessary optimality conditions.

b) Consider the precise set $C = \{x \in \mathbb{R}^n \colon a_i \leq x_i \leq b_i,\ i = 1, \dots, n\}$ with real numbers $a_i < b_i$, $i = 1, \dots, n$. State the projection of a vector $y \in \mathbb{R}^n$ on this set C explicitly.