Numerical Simulation

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Lecturer: Prof. Dr. Ira Neitzel
Assistant: Dr. Guanglian Li

universitätbonn

Exercise Sheet 2

Closing date **April 26, 2016**.

Exercise 1. (A finite-dimensional control-constrained optimal control problem)

Consider the control problem

$$\min_{u\in\mathbb{R}^m} \frac{1}{2}|y - y_d|^2 + \frac{\lambda}{2}|u|^2$$

s.t.

$$Ay = Bu, \quad u \le u_b,$$

where $A \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $B \in \mathbb{R}^{n\times m}$, $y_d \in \mathbb{R}^n$, $u_b \in \mathbb{R}^m$, and $\lambda \in \mathbb{R}^+$ are given. Prove existence and uniqueness of a solution $\bar{u} \in \mathbb{R}^m$ with associated state $\bar{y} \in \mathbb{R}^n$, and show that

$$\bar{u}_i = \begin{cases} u_{b,i} & \text{if } (\lambda\bar{u} + B^T\bar{p})_i < 0 \\ -\frac{1}{\lambda}\bar{p}_i & \text{else.} \end{cases}$$

Here, $\bar{p}$ denotes the adjoint state as discussed in the lecture. You may want to prove and explore the fact that $\bar{u}$ solves the linear optimization problem

$$\min_{u\in\mathbb{R}^m} (B^T\bar{p} + \lambda\bar{u}, u)_{\mathbb{R}^m}$$

s.t.

$$u \le u_b.$$

All inequalities have to be understood componentwise.

(6 points)

Exercise 2. (Properties of projections)

Let $C \subset \mathbb{R}^n$ be a nonempty convex and closed set. The projection $x = P_C(y)$ of a vector $y \in \mathbb{R}^n$ on the set C is defined as solution of: $\min_{x\in C} \frac{1}{2}\|x - y\|_2^2$.

Prove that for arbitrary $y_1, y_2 \in \mathbb{R}^n$ the following holds:

a) $(P_C(y_1) - P_C(y_2))^T(y_1 - y_2) \ge \|P_C(y_1) - P_C(y_2)\|_2^2$, i.e. P_C is monotone,

b) $\|P_C(y_1) - P_C(y_2)\|_2 \le \|y_1 - y_2\|_2$, i.e. P_C is non-expanding.

(4 points)

Exercise 3. (Finite-dimensional optimization problem with linear inequality constraints)

Solve the problem

$$\min_{x\in\mathbb{R}^2} f(x_1, x_2) := (x_1 - 1)^2 + (x_2 - 2)^2$$

s.t.

$$\begin{aligned} g_1(x) &:= x_1 + x_2 - 1 \le 0 \\ g_2(x) &:= -x_1 \le 0 \\ g_3(x) &:= -x_2 \le 0 \end{aligned}$$

with the help of the Lagrange technique: First, prove existence and uniqueness of a solution, and state the KKT-system for the above problem. Then, find KKT-points (i.e. solutions of the KKT-sytem, including Lagrange multipliers). Is the point $\bar{x}$ you found a solution of the optimization problem?

Note: It can be shown that any point

$$x \in X := \{x \in \mathbb{R}^n : g_i(x) \le 0, \quad i = 1, 2, 3\}$$

fulfills a so-called **constraint qualification** (you do not need to prove this), the existence of Lagrange multipliers is indeed a necessary optimality condition.

(10 points)

Exercise 4. We are looking for a point x in the real plane $\mathbb{R}^2$, such that the sum of the distances of x to three given points $x_1, x_2, x_3 \in \mathbb{R}^2$ is minimal.

a) Formulate this problem as a minimization problem in $\mathbb{R}^2$. Prove existence of a solution x^*. Is it unique?

b) Let $x^* \neq x_i$, $i = 1 \ldots 3$ be given. Use the first order necessary optimality conditions and a graphical sketch of the problem to find x^* **inside** the triangle x_1, x_2, x_3. Prove that then all interior angles of this triangle x_1, x_2, x_3 are less than 120^{o}.

c) What happens, if one of these interior angles is larger than $> 120^{\text{o}}$?

(7 points)