Numerical Simulation

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Exercise Sheet 3

Closing date **May 3, 2016**.

Exercise 1. Consider the optimization problem

$$\min\, J(u) := \frac{1}{2}\int_0^1 x^2 u'(x)^2 dx - \int_0^1 u(x)dx, \quad u \in H_0^1(I), \quad I = (0,1),$$

and prove that it does not admit a solution.
Clue: Prove first that

$$J(u) \geq -\frac{1}{2} \quad \forall u \in H_0^1(I),$$

then construct a sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \in H_0^1(I)\, \forall n \in \mathbb{N}$ with

$$J(u_n) \to -\frac{1}{2}, \quad n \to \infty$$

and prove that there is no function $u \in H_0^1(I)$ with

$$J(u) = -\frac{1}{2}.$$

(10 points)

Exercise 2. Consider $\Omega = (0,1)$ and the optimal control problem

$$\begin{aligned} \min_{(u,y)\in L^2(\Omega)\times V} \quad & J(u,y) = \tfrac{1}{2}\|y-1\|^2_{L^2(\Omega)} \\ -y'' &= u \quad \text{in } \Omega, \\ y'(1) &= 0 \\ y(0) &= 0. \end{aligned}$$

What is an appropriate choice for the state space V and the set of admissible pairs $(u,y) \in W_{\mathrm{ad}}$? Prove that

$$j := \inf_{(u,y)\in W_{\mathrm{ad}}} J(y,u) = 0,$$

but the problem does not admit a global solution.

(10 points)

Exercise 3. Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\beta = (\beta^1, \beta^2) \in H^1(\Omega)\times H^1(\Omega)$, and $y \in H_0^1(\Omega)$.

a) Prove that the following integral is finite:

$$\int_\Omega \beta(x)\cdot \nabla y(x) y(x) dx.$$

Use embedding theorems and Hölder's inequality.

b) Prove that

$$\int_\Omega \beta(x) \cdot \nabla y(x) y(x) dx = -\frac{1}{2} \text{div}\, \beta(x) y(x)^2 dx.$$

c) Consider the boundary value problem

$$\begin{cases} -\Delta y + \beta \cdot \nabla y + cy &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \end{cases}$$

with $f \in L^2(\Omega)$ and $c \in \mathbb{R}$, $c > 0$. Derive a condition on β and c that is sufficient for existence and uniqueness of a weak solution.

(10 points)