Numerical Simulation

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Exercise Sheet 5

Closing date **May 24, 2016**.

Exercise 1. a) Find all solutions to

$$\min_{u\in L^4(0,1)} f(u) := \int_0^1 \left(u^2(x)-1\right)^2 \, dx \quad \text{s.t. } |u(x)| \leq 1 \text{ a.e. in } (0,1)$$

b) Find a Banach space where all solutions to a) have the same distance.

c) Show that the directional derivative $f'(u)h$ exists for all $u, h \in L^4(0,1)$. Is f Fréchet differentiable in $L^4(0,1)$? In the Banach space stated in b)?

(10 points)

Exercise 2. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded Lipschitz domain, $y_\Omega \in L^2(\Omega)$, $y_\gamma \in L^2(\Gamma)$ and $\beta \in L^\infty(\Omega)$ as well as $0 \leq \alpha \in L^\infty(\Gamma)$ with $\|\alpha\|_{L^\infty(\Gamma)} > 0$ be given functions, and $\lambda, \lambda_\Omega, \lambda_\Gamma > 0$ be positive real numbers. Moreover, let $u_a, u_b \in L^\infty(\Omega)$ with $u_a(x) \leq u_b(x)$ a.e. be given, and define

$$U_{\text{ad}} := \{u \in L^2(\Omega) \colon u_a(x) \leq u(x) \leq u_b(x) \quad \text{a.e. in } \Omega\}.$$

State and prove the first order necessary and sufficient optimality conditions for the problem

$$\min_{(u\in U_{\text{ad}}), y\in H^1(\Omega)} J(y,u) = \tfrac{\lambda_\Omega}{2}\|y-y_d\|^2_{L^2\Omega} + \tfrac{\lambda_\Gamma}{2}\|y-y_\Gamma\|^2_{L^2\Gamma} + \lambda\|u\|^2_{L^2(\Omega)}$$

$$\begin{aligned} -\Delta y &= \beta u \quad \text{in } \Omega \\ \partial_\nu y + \alpha y &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $\partial_\nu y$ denotes the outward unit normal of y.

(10 points)

Exercise 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $e \in L^2(\Omega)$ a given function, and for the control space $U = \mathbb{R}$ let $S \colon U \to L^2(\Omega)$ denote the control-to-state operator for the state equation

$$\begin{aligned} -\Delta y &= u\, e \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Calculate $S^*z \in \mathbb{R}$ for given $z \in L^2(\Omega)$.

(10 points)