Wissenschaftliches Rechnen II/Scientific Computing II

Sommersemester 2016
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Exercise sheet 1: Warm up

To be handed in on **Thursday, 21.4.2016**

This exercise sheet is a warm up. The exercises are intended to recall some concepts which you should have seen before during your studies and which will help you understand the contents of this course.

1 Hilbert spaces: basic notions

Definition 1 (Scalar product, pre-Hilbert space)**.** Let E be a vector space over $\mathbb{R}$. The map $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{R}$ is called *scalar product* or *inner product* on E if

(S1) $x \mapsto \langle x, y\rangle$ is linear for all $y \in E$,

(S2) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in E$.

(S3) $\langle x, x\rangle \geq 0$ for all $x \in E$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

The pair $(E, \langle\cdot,\cdot\rangle)$ or simply E is called a *pre-Hilbert space*.

Every pre-Hilbert space $(E, \langle\cdot,\cdot\rangle)$ is a normed space with norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Definition 2. A complete pre-Hilbert space is called a *Hilbert space*.

Definition 3. Let E be a pre-Hilbert space. A countable subset $S \subset E$ is called *orthonormal system* (ONS) if for all $x, y \in E$ we have

$$\langle x, y\rangle = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

A orthonormal system S is called *orthonormal basis* (ONB) if span S is dense in E.

2 Group work

Please work on the following exercises in groups of three to four people during the exercise session. The tutor can help you with hints if you have problems solving the exercises.

G 1. (Cauchy-Schwarz inequality)

Let E be a pre-Hilbert space over $\mathbb{R}$ with inner product $\langle\cdot,\cdot\rangle$. Show that for all $x, y \in E$

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle.$$

Hint: Before you try to prove it, make sure you understand the geometrical meaning of the above inequality.

G 2. (The mother of all Hilbert spaces)

Consider the sequence space

$$\ell_2(\mathbb{N}) = \{(x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}.$$

Prove the following claims:

a) The map $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ defines an inner product on $\ell_2(\mathbb{N})$.

b) The space $\ell_2(\mathbb{N})$ with inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space.

G 3. (Bessel's inequality)

Let E be a pre-Hilbert space and $(e_i)_{i\in\mathbb{N}}$ be a ONS in E. Show that for all $x \in E$

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

Discuss the geometrical meaning of the inequality. **Hint:** Compare with G 1.

G 4. Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and further, let S be a ONS in H. Prove that the following statements are equivalent:

1. S is a ONB.
2. For $x, y \in H$ we have $\langle x, y\rangle = \sum_{e\in S} \langle x, e\rangle\langle e, y\rangle$.
3. For all $x \in H$ we have $\|x\|^2 = \sum_{e\in S} |\langle x, e\rangle|^2$.
4. $S^{\perp} = \{x \in H : \langle x, e\rangle = 0 \text{ for all } e \in S\} = \{0\}$.
5. S is a maximal ONS, that is, there is no ONS S' such that $S \subsetneq S'$.
6. $\operatorname{span} S$ is dense in H.

3 Homework

The following exercises are homework. Throughout this course, you are supposed to solve and hand in the homework as a group of two people. Use the first session to find a partner. Since this exercise sheet is only a warm up this homework will not be graded. But if you hand it in we will correct it for you.

H 1. (Riesz representation theorem)

Consider the Hilbert space $\ell_2(\mathbb{N})$ introduced in G 2.

a) Show that for any $n \in \mathbb{N}$, the map $\mu_1 : \ell_2(\mathbb{N}) \to \mathbb{R}, x \mapsto \sum_{i=1}^{n} x_i$ is a continuous linear functional on $\ell_2(\mathbb{N})$. What element $u \in H$ represents μ_1?

b) Consider the map $\mu_2(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i$. Show that the map is well-defined on $\ell_2(\mathbb{N})$ and show that it is a continuous linear functional. What is its Riesz representer?

H 2. (Newton interpolation)

Assume to be given data points $(x_0, y_0), \dots, (x_n, y_n) \in \mathbb{R}^2$, such that the x_i are pairwise different. We are seeking a polynomial P of degree n which interpolates the given data points, that is

$$P(x_j) = y_j \quad \text{for } j = 1, \dots, n.$$

If P takes the form

$$P(x) = \sum_{i=0}^{n} c_i N_i(x),$$

where $c_1, \dots, c_n \in \mathbb{R}$ and the N_i are *Newton basis polynomials*,

$$N_0(x) = 1, \quad N_i(x) = \prod_{j=0}^{i-1} (x - x_j),$$

then P is called *interpolation polynomial in Newton form.*

a) Show that if the x_i are pairwise different, then the interpolation problem

$$\begin{pmatrix} N_0(x_0) & \cdots & N_n(x_0) \\ \vdots & \ddots & \vdots \\ N_0(x_n) & \cdots & N_n(x_n) \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$$

always has a solution and this solution is unique.

b) It can be shown that the coefficients $c_1, \dots, c_n$ are given by the *divided differences*

$$c_i = [x_1, \dots, x_i]f.$$

These divided differences are recursively defined by

$$\begin{aligned} [x_i]f &= y_i \\ [x_i, \dots, x_j]f &= \frac{[x_{i+1}, \dots, x_j]f - [x_i, \dots, x_{j-1}]f}{x_j - x_i} \qquad i < j. \end{aligned}$$

Determine the coefficient $c_1, \dots, c_n$ of the interpolation polynomial in Newton form for the following data points:

i	0	1	2	3	4
x_i	-5	-2	-1	0	1
y_i	17	8	21	42	35

H 3. (Programming exercise)

This exercise assumes you have already a working python installation including the following packages: `numpy`, `matplotlib`, `jupyter`. See the course homepage for information how to get a working python installation.

a) Download *sheet1-notebook.ipynb* from the course homepage.

b) Start `jupyter notebook` from a terminal. Import *sheet1-notebook.ipynb* and replace the title with *[Your name] Notebook 1.*

c) Solve the tasks posed to you in the notebook.

d) If you are done, mail the notebook file to `mayer@ins.uni-bonn.de`.