Exercise sheet 12 To be handed in on **Thursday, 14.07.2016**

Isomap

1 Group exercises

G 1. Consider the manifold $M = \{x \in \mathbb{R}^d : \|x\|_2 = r\}$, i.e., the Euclidean sphere with radius $r > 0$ in $\mathbb{R}^d$. Compute the minimum radius of curvature $r_0(M)$ and the minimum branch separation $s_0(M)$. Moreover, show that Lemma 2.16 holds true with equality, i.e., for all $x, y \in M$ such that $d_M(x, y) < \pi r_0(M)$ we have

$$d_E(x, y) = 2r_0(M) \sin\left(\frac{d_M(x, y)}{2r_0(M)}\right).$$

G 2. Construct a graph distance matrix which is not a Euclidean distance matrix.

G 3. Let $M \subset \mathbb{R}^d$ be manifold. Assume you have run Isomap with input $x_1, \ldots, x_n \in M$ to obtain p-dimensional embeddings. Write down ready-to-use formulas that compute you the p-dimensional embedding for a new, unseen data point $x \in M$ without rerunning the whole Isomap algorithm.

2 Homework

H 1. Let M be a compact manifold. Prove the following simplyfied version of Lemma 2.16. For any $\varepsilon > 0$, we have

$$(1 - \varepsilon) d_M(x, y) \leq d_E(x, y) \leq d_M(x, y)$$

for all $x, y \in M$ such that $d_M(x, y) < 2r_0(M)\varepsilon$. **Hint:** Consider a unit speed parametrization $\gamma : (0, l) \to M$ with $\gamma(0) = x$, $\gamma(l) = y$ and use the fundamental theorem of calculus to obtain a first order estimate.

(10 Punkte)

H 2. (Isomap and non-EDM graph distance matrices)

a) Let S be a symmetric $n \times n$ matrix with eigenvalue decomposition $U \Lambda U^T$. Let Λ' be Λ with all negative eigenvalues replaced by zero and put $S^+ := U \Lambda' U^T$. Show that S^+ is the solution of
$$\min_{B \in \mathcal{S}_n^+} \|S - B\|_F^2,$$
where $\mathcal{S}_n^+$ is the set of all positive semi-definite $n \times n$ matrices.

b) Based on your insights from a), argue why Isomap can also be used when the graph distance matrix D_G is not a Euclidean distance matrix. How should the algorithm be modified in this case?

(10 Punkte)