Wissenschaftliches Rechnen II/Scientific Computing II

Sommersemester 2016
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Exercise sheet 2

To be handed in on **Thursday, 28.4.2016**

1 Group exercises

G 1. (Fourier system)

Consider the Hilbert space

$$L_2([-\pi,\pi]) = \left\{ f : [-\pi,\pi] \to \mathbb{C} \text{ such that } \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\}$$

with inner product $\langle f, g\rangle = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$. Show that the *Fourier system* $(e_n)_{n\in\mathbb{N}}$ given by $e_n(x) = \exp(inx)$ forms an orthonormal system in $L_2([-\pi,\pi])$.

Solution. Simple calculations:

$$\int_{-\pi}^{\pi} e^{inx}\overline{e^{inx}}dx = \int_{-\pi}^{\pi} e^{inx}e^{-inx}dx = 2\pi.$$

$$\begin{aligned}\int_{-\pi}^{\pi} e^{i(n-m)x} &= \int \cos((n-m)x)dx + i\int \sin((n-m)x)dx \\ &= \left[\frac{\sin((n-m)x)}{n-m}\right]_{-\pi}^{\pi} - i\left[\frac{\cos((n-m)x)}{n-m}\right]_{-\pi}^{\pi} = 0.\end{aligned}$$

□

G 2. (Kernel spaces of trigonometric polynomials, Dirichlet kernel)

For $n \in \mathbb{N}$ consider the space of trigonometric polynomials

$$\mathcal{H}_n := \left\{ f : [-\pi,\pi] \to \mathbb{C} \;:\; f(x) = \sum_{k=-n}^{n} \alpha_k \exp(ikx),\ \alpha_k \in \mathbb{C} \right\}.$$

equipped with the L_2-inner product $\langle\cdot,\cdot\rangle$ defined in Exercise G1. Show that $\mathcal{H}_n$ is a reproducing kernel Hilbert space with kernel $D_n(x,y) = 1 + 2\sum_{k=1}^{n}\cos(k(x-y))$. Is D_n well-defined for $n\to\infty$? **Remark:** D_n is called *Dirichlet kernel.*

Solution. Let $f(x) = \sum_{k=-n}^{n}\alpha_k e_k(x) \in \mathcal{H}_n$. Using 1, we know that $\langle f, e_k\rangle = \alpha_k$. Hence,

$$f(x) = \sum_{k=-n}^{n}\langle f, e_k\rangle e_k(x) = \langle f, \sum_{k=-n}^{n}\overline{e_k(x)}e_k\rangle = \langle f, D_n(\cdot,x)\rangle$$

with

$$D_n(y,x) = \sum_{k=-n}^{n} e^{-ikx}e^{-iky} = \sum_{k=-n}^{n} e^{ik(y-x)}$$
$$= \sum_{k=-n}^{n} \cos(k(y-x)) + i\sum_{k=-n}^{n} \sin(k(y-x)) = 1 + 2\sum_{k=1}^{n} \cos(k(y-x)).$$

We have $\lim_{n\to\infty} D_n(0,0) = \infty$, hence the point-wise limit does not exist. **Remark:** One can even show that D_n does not converge in $L_1([-\pi,\pi])$. □

G 3. (Fejér kernels)

Let $f \in L_2([-\pi,\pi])$. The nth *Fourier partial sum* is given by

$$s_n(\theta) := s_n(f)(\theta) := \sum_{k=-n}^{n} \langle f, e_k\rangle e_k(\theta) = \frac{1}{2\pi}\sum_{k=-n}^{n} e_k(\theta)\int_{-\pi}^{\pi} f(x)e_k(-x)dx,$$

and $\langle f, e_k\rangle$ is called the kth *Fourier coefficient*. The nth *Cesàro* mean is given by $\sigma_n(\theta) := \sigma_n(f)(\theta) := \frac{1}{n+1}\sum_{k=0}^{n} s_k(\theta)$.

a) By considering the Fourier coefficients of s_n and σ_n, discuss the difference between approximating with Fourier partial sums and approximating with Cesàro means.

b) Show that $\sigma_n(\theta) = \langle f, \phi_n(\theta,\cdot)\rangle$, where

$$\phi_n(x,y) = \frac{1}{n+1}\left(\frac{\sin((n+1)(x-y)/2)}{\sin((x-y)/2)}\right)^2.$$

Hint: Use the *trigonometric identity* $\sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+1/2)x)}{\sin(x/2)}$.

c) Determine the Hilbert space $\mathcal{H}(\phi_n) \subset L_2([-\pi,\pi])$ such that $\phi_n(x,y)$ is the reproducing kernel. By comparing the unit balls of $\mathcal{H}(\phi_n)$ and $\mathcal{H}_n$ (see G2), which Hilbert space contains the "smoother" trigonometric polynomials in the sense that high oscillations are more penalized?

Solution.

a) For $k \in \{-n,\dots,0,\dots,n\}$ we have $\langle \sigma_n, e_k\rangle = \frac{n+1-|k|}{n+1}\langle f, e_k\rangle$. Hence, in the Fourier partial sum every frequency contributes equally to the approximation, whereas in the Cesàro mean the higher frequencies play a less and less important role.

b) Observe that $\sigma_n(\theta) = \frac{1}{n+1}\sum_{k=0}^{n}\langle f, D_k(\cdot,\theta)\rangle = \langle f, \frac{1}{n+1}\sum_{k=0}^{n} D_k(\cdot,x)\rangle$. Using the trigonometric identity,

$$\phi_n(y,x) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(y,x) = \frac{1}{n}\sum_{k=0}^{n-1} e_k(y-x)$$
$$= \frac{1}{n}\sum_{k=0}^{n-1}\frac{\sin(k+1/2)(y-x)}{\sin((y-x)/2)} = \frac{1}{n\sin((y-x)/2)}\Im\sum_{k=0}^{n-1} e^{i(k+1/2)(x-y)}$$
$$= \frac{1}{n}\left(\frac{\sin(n(y-x)/2)}{\sin((y-x)/2)}\right)^2.$$

To derive the last line we have summed the the geometric series, $\sum_{k=0}^{n-1} a_0 q^k = a_0\frac{1-q^n}{1-q}$.

c) Consider the space of trigonometric polynomials $\mathcal{H}_n$ from G2. Let $\mathcal{H}(\phi_n)$ be $\mathcal{H}_n$ equipped with the inner product

$$\langle f, g\rangle_{\phi_n} := \sum_{k=-n}^{n} \frac{n+1}{n+1-|k|}\langle f, e_k\rangle\overline{\langle g, e_k\rangle}.$$

Now observe that $\langle f, \phi_n(\cdot,\sigma)\rangle = \sum_{k=-n}^{n} \frac{n+1}{n+1-|k|}\langle f, e_k\rangle\overline{\langle \phi_n(\cdot,\theta), e_k\rangle}$ Since $\overline{\langle \phi_n(\cdot,\theta), e_k\rangle} = \langle e_k, \phi_n(\cdot,\theta)\rangle = \sigma_n(e_k)(\theta)$, we get

$$\langle f, \phi_n(\cdot,\sigma)\rangle_{\phi_n} = \sigma_n\left(\sum_{k=-n}^{n} \frac{n+1}{n+1-|k|}\langle f, e_k\rangle e_k\right)(\theta) = \sum_{k=-n}^{n}\langle f, e_k\rangle e_k(\theta) = f(\theta).$$

Comparing the scalar products, we see that high oszillations are more penalized in $\mathcal{H}(\phi_n)$ than in $\mathcal{H}_n$.

$\square$