Exercise sheet 2

To be handed in on **Thursday, 28.4.2016**

1 Group exercises

G 1. (Fourier system)

Consider the Hilbert space

$$L_2([-\pi,\pi]) = \left\{ f : [-\pi,\pi] \to \mathbb{C} \text{ such that } \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\}$$

with inner product $\langle f, g\rangle = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$. Show that the *Fourier system* $(e_n)_{n\in\mathbb{N}}$ given by $e_n(x) = \exp(inx)$ forms an orthonormal system in $L_2([-\pi,\pi])$.

G 2. (Kernel spaces of trigonometric polynomials, Dirichlet kernel)

For $n \in \mathbb{N}$ consider the space of trigonometric polynomials

$$\mathcal{H}_n := \left\{ f : [-\pi,\pi] \to \mathbb{C} \ :\ f(x) = \sum_{k=-n}^{n} \alpha_k \exp(ikx),\ \alpha_k \in \mathbb{C} \right\}.$$

equipped with the L_2-inner product $\langle\cdot,\cdot\rangle$ defined in Exercise G1. Show that $\mathcal{H}_n$ is a reproducing kernel Hilbert space with kernel $D_n(x,y) = 1 + 2\sum_{k=1}^{n}\cos(k(x-y))$. Is D_n well-defined for $n\to\infty$? **Remark:** D_n is called *Dirichlet kernel.*

G 3. (Fejér kernels)

Let $f \in L_2([-\pi,\pi])$. The nth *Fourier partial sum* is given by

$$s_n(\theta) := s_n(f)(\theta) := \sum_{k=-n}^{n} \langle f, e_k\rangle e_k(\theta) = \frac{1}{2\pi}\sum_{k=-n}^{n} e_k(\theta)\int_{-\pi}^{\pi} f(x) e_k(-x)dx,$$

and $\langle f, e_k\rangle$ is called the kth *Fourier coefficient*. The nth *Cesàro* mean is given by $\sigma_n(\theta) := \sigma_n(f)(\theta) := \frac{1}{n+1}\sum_{k=0}^{n} s_k(\theta)$.

a) By considering the Fourier coefficients of s_n and σ_n, discuss the difference between approximating with Fourier partial sums and approximating with Cesàro means.

b) Show that $\sigma_n(\theta) = \langle f, \phi_n(\theta,\cdot)\rangle$, where

$$\phi_n(x,y) = \frac{1}{n+1}\left(\frac{\sin((n+1)(x-y)/2)}{\sin((x-y)/2)}\right)^2.$$

Hint: Use the *trigonometric identity* $\sum_{k=-n}^{n} e^{ikx} = \frac{\sin((n+1/2)x)}{\sin(x/2)}$.

c) Determine the Hilbert space $\mathcal{H}(\phi_n) \subset L_2([-\pi,\pi])$ such that $\phi_n(x,y)$ is the reproducing kernel. By comparing the unit balls of $\mathcal{H}(\phi_n)$ and $\mathcal{H}_n$ (see G2), which Hilbert space contains the “smoother” trigonometric polynomials in the sense that high oszillations are more penalized?

2 Homework

H 1. This homework exercise deals with the concept of *approximate identities* which is conceptually strongly related to reproducing kernels.

Definition 1 (Approximate identity)**.** A sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ over $[-\pi,\pi]$ is called *approximate identity* if

1. $\phi_n \geq 0$.
2. $\int_{-\pi}^{\pi} \phi_n(x)\mu(dx) = 2\pi$.
3. For each $\delta > 0$, we have $\lim_{n\to\infty} \int_{\{|x|\leq\delta\}} \phi_n(x)\mu(dx) = 2\pi$.

   Equivalently, for each $\delta > 0$, $\lim_{n\to\infty} \int_{\{|x|>\delta\}} \phi_n(x)\mu(dx) = 0$.

Let $(\phi_n)_{n\in\mathbb{N}}$ be the sequence of Fejér kernels introduced in G 3.

a) With the abuse of notation $\phi_n(x) = \phi_n(0,-x)$, show that $(\phi_n)_{n\in\mathbb{N}}$ is an approximate identity.

b) Plot the Dirichlet and Fejér kernels for various values of n. Use the accompanying Jupyter notebook which you find on the lectures' homepage.

c) Let $\mathcal{C}$ be the set of all continuous, 2π-periodic functions $f : [-\pi,\pi] \to \mathbb{C}$. Using a) show that $\operatorname{span}\{e_k : k \in \mathbb{N}\}$ is dense in $\mathcal{C}$ with respect to the uniform norm $\|f\|_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|$.

d) Show that the Fouriersystem introduced in G1 is an orthonormal Hilbertbasis in $L_2([-\pi,\pi])$. **Hint:** Use the well-known fact that the set of continous, 2π-periodic functions is dense in $L_2([-\pi,\pi])$.

(8 Punkte)

H 2. (Structure implied by the kernel)

Let Ω be some set and $\mathcal{H}$ be a Hilbert space of real valued functions on Ω with continuous point evaluations. Let $k : \Omega \times \Omega \to \mathbb{R}$ denote the reproducing kernel of $\mathcal{H}$.

a) Show that $d : \Omega\times\Omega \to [0,\infty]$, $d(x,y) := \sqrt{k(x,x) + k(x,y) - 2k(x,y)}$ defines a *pseudometric* on Ω, that is,

$$\begin{aligned} (i) \quad & d(x,y) = d(y,x) \\ (ii) \quad & d(x,y) \leq d(x,z) + d(y,z). \end{aligned}$$

b) Assume that $\mathcal{H}$ *separates* the points of Ω, that is, for every pair of different points $x, y \in \Omega$, $x \neq y$, there is a function $f \in \mathcal{H}$ such that $f(x) \neq f(y)$. Show that d is then even a metric on Ω.

c) Assume again that $\mathcal{H}$ separates the points of Ω. Show that all functions $f \in \mathcal{H}$ are always Lipschitz-continuous with respect to d.

d) Let $\Omega = \mathbb{R}$. Assume that the Hilbert space $\mathcal{H}$ is such that the *Gauß kernel* $k(x,y) := \exp(-|x-y|)$ is the reproducing kernel. Show that $\mathcal{H}$ separates the points of Ω.

(6 Punkte)

H 3. (Interpolation with the Gauss kernel)

This homework deals with the interpolation of 2-variate functions using the Gauss kernel

$$k(x,y) = \exp(-\gamma\|x-y\|_2^2)\ .$$

Please use the accompanying Jupyter notebook to solve the programming tasks. You find the notebook on the lecture's homepage.

(6 Punkte)