# Exercise sheet 4 - Sample solutions for group exercises

**G 1.** Let $k : \Omega^2 \to \mathbb{R}$ be positive semi-definite. Provide details for the proof of Theorem 34. Concretely,

a) Recall $H = \operatorname{span}\{k(x,\cdot) \mid x \in \Omega\}$ with inner product $\langle \cdot, \cdot \rangle_H$ given by

$$\langle f, g, \rangle H = \sum_{i=1}^{n} \sum_{j=}^{m} a_i b_j k(x_i, y_j)$$

for $f = \sum_{i=1}^{n} a_i k(x_i, \cdot)$, $g = \sum_{j=1}^{m} b_j K(y_j, \cdot)$. Let $(h_n)_{n\in\mathbb{N}}$, $h_n \in H$, be a Cauchy sequence. Show that the pointwise limit $h(t) := \lim_{n\to\infty} h_n(t)$ exists.

b) Let $\mathcal{N}_k$ be the set of pointwise limits of arbitrary Cauchy sequences in $H$. For $g, f \in \mathcal{N}_k$, let

$$\langle f, g \rangle_k := \lim_{n\to\infty} \langle f_n, g_n \rangle_H.$$

Verify that $\langle f, f \rangle = 0$ if and only if $f = 0$, that is, $\langle \cdot, \cdot \rangle$ is indeed a scalar product.

c) Show that $\mathcal{N}_k$ is complete with respect to the norm induced by $\langle \cdot, \cdot \rangle_k$.

*Solution.*

a) Since

$$|h_n(x) - h_m(x)| = |\langle h_n - h_m, k(x, \cdot), \rangle H| \le \|h_n - h_m\|_H \|k(x, \cdot)\|_H = \|h_n - h_m\|_H \sqrt{k(x,x)}$$

the pointwise limit $h(t) := \lim_{n\to\infty} h_n(t)$ exists.

b) Let $f$ be the pointwise limit of a Cauchy sequence $(f_n)_{n\in\mathbb{N}}$ in $H$. Assume $f = 0$. For fixed $m \in \mathbb{N}$, consider the element $f_l$, which can be written as

$$f_l = \sum_{i=1}^{m} a_i k(x_i, \cdot).$$

Then $\lim_{n\to\infty} \langle f_n, f_l \rangle_H = \lim_{n\to\infty} \sum_{i=1}^{m} a_i f_n(x_i) = 0$. Now for $\varepsilon > 0$, the estimate

$$\langle f_l, f_l \rangle_H \le |\langle f_n, f_l \rangle| + \|f_l\|_H \|f_n - f_l\|_H$$

yields $\|f_l\|_H^2 = \langle f_l, f_l \rangle_H \le \varepsilon \|f_l\|_H$ if we choose $l$ sufficiently large and let $n \to \infty$. Since $\varepsilon$ can be choosen arbitrarily small, we get $\langle f, f \rangle_k = \lim_{l\to\infty} \langle f_l, f_l \rangle_H = 0$. Now assume $\langle f, f \rangle_k = 0$. From the reproducing property

$$\langle f, k(x, \cdot) \rangle = \lim_{n\to\infty} \langle f_n, k(x, \cdot) \rangle = \lim_{n\to\infty} f_n(t) = f(t)$$

we get $|f(t)| \le \langle f, f \rangle_H \sqrt{k(x,x)} = 0$.

c) For $\varepsilon > 0$ let $n_0, m \in \mathbb{N}$ be such that $\|f_{n_0} - f_m\|_h < \varepsilon$. But then

$$\|f - f_m\|_k^2 = \langle f - f_m, f - f_m\rangle_k = \lim_{n\to\infty} \langle f_n - f_m, f_n - f_m\rangle_H = \lim_{n\to\infty} \|f_n - f_m\|_H^2 < \varepsilon^2.$$

Since we can choose $\varepsilon$ arbitrarily small we may conclude that $(f_n)_n$ converges to $f$ with respect to the norm $\|\cdot\|_k$.

□

**G 2.** (Interpolation and discrete Fourier transform)

For even $n \in \mathbb{N}$, consider again the space of complex-valued trigonometric polynomials $\mathcal{H}_n$ introduced in Sheet 2, Exercise G2. Further consider the sampling points

$$x_k = \frac{2\pi k}{n} \in [-\pi, \pi], \quad k = -n/2, \ldots, 0, \ldots, n/2 - 1.$$

a) Consider the subspace $\widetilde{\mathcal{H}}_n = \{f \in \mathcal{H}_n : \langle f, e_0\rangle = 0\}$, which has the kernel $\widetilde{D}_n(x, y) = 2\sum_{k=1}^n \cos(k(x - y))$. Show that the matrix $\widetilde{\mathbf{D}}_n := \frac{1}{2n}(\widetilde{D}_n(x_j, x_l))_{j,l=-n/2,\ldots,n/2-1}$ is the identity, that is,

$$\widetilde{D}_n(2\pi j/n, 2\pi l/n) = \begin{cases} 2n & : j = l \\ 0 & : j \neq l. \end{cases}$$

**Hint:** Use the following fact about geometric series: for $r \neq 1$, we have $\sum_{k=0}^n r^k = \frac{1-r^{n+1}}{1-r}$.

b) Assume to be given observations $(x_k, y_k)_{k=-n/2,\ldots,n/2-1}$. Determine the polynomial $g_n \in \widetilde{H}_n$ which solves the interpolation problem

$$g_n(x_i) = y_i, \quad i = -n/2, \ldots, n/2 - 1.$$

Determine the Fourier coefficients of $g_n$.

**Remark:** The Fourier coefficients of $g_n$ are the *discrete Fourier transform* of the vector $(y_k)_{k=-n/2,\ldots,n/2-1}$.

*Solution.*

a) The case $j = l$ is obvious, so assume $j \neq l$ and write $\Delta = j - l$. Then,

$$\begin{aligned}
\widetilde{D}_n(2\pi j/n, 2\pi l/n) &= 2\sum_{k=1}^n \cos(2\pi k\Delta/n) \\
&= 2\Re\sum_{k=1}^n e^{2\pi ik\Delta/n} = 2\Re\sum_{k=0}^n e^{2\pi ik\Delta/n} - 2 \\
&= 2\Re\frac{1 - e^{2\pi i(n+1)\Delta/n}}{1 - e^{2\pi ik\Delta/n}} - 2 = 0
\end{aligned}$$

b) According to a) the interpolation polynomial is simply given by

$$g_n = \frac{1}{2n}\sum_{i=-n/2}^{n/2-1} y_i D_n(x_i, \cdot)$$

For $k \neq 0$, we have $\langle D_n(x_i), e_k\rangle = \overline{\langle e_k, D_n(x_i)\rangle} = \overline{e_k(x_i)} = e_k(-x_i)$ and thus

$$\langle g_n, e_k\rangle = \frac{1}{2n}\sum_{i=-n/2}^{n/2-1} y_i\langle D_n(x_i), e_k\rangle = \frac{1}{2n}\sum_{i=-n/2}^{n/2-1} y_i e_k(-x_i).$$

□