Wissenschaftliches Rechnen II/Scientific Computing II

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Exercise sheet 4

To be handed in on **Thursday, 12.05.2016**

1 Group exercises

G 1. Let $k : \Omega^2 \to \mathbb{R}$ be positive semi-definite. Provide details for the proof of Theorem 34. Concretely,

a) Recall $H = \operatorname{span}\{k(x, \cdot) \mid x \in \Omega\}$ with inner product $\langle \cdot, \cdot \rangle_H$ given by

$$\langle f, g, \rangle H = \sum_{i=1}^{n} \sum_{j=}^{m} a_i b_j k(x_i, y_j)$$

for $f = \sum_{i=1}^{n} a_i k(x_i, \cdot)$, $g = \sum_{j=1}^{m} b_j K(y_j, \cdot)$. Let $(h_n)_{n\in\mathbb{N}}$, $h_n \in H$, be a Cauchy sequence. Show that the pointwise limit $h(t) := \lim_{n\to\infty} h_n(t)$ exists.

b) Let $\mathcal{N}_k$ be the set of pointwise limits of arbitrary Cauchy sequences in H. For $g, f \in \mathcal{N}_k$, let

$$\langle f, g \rangle_k := \lim_{n\to\infty} \langle f_n, g_n \rangle_H.$$

Verify that $\langle f, f \rangle = 0$ if and only if $f = 0$, that is, $\langle \cdot, \cdot \rangle$ is indeed a scalar product.

c) Show that $\mathcal{N}_k$ is complete with respect to the norm induced by $\langle \cdot, \cdot \rangle_k$.

G 2. (Interpolation and discrete Fourier transform)

For even $n \in \mathbb{N}$, consider again the space of complex-valued trigonometric polynomials $\mathcal{H}_n$ introduced in Sheet 2, Exercise G2. Further consider the sampling points

$$x_k = \frac{2\pi k}{n} \in [-\pi, \pi], \quad k = -n/2, \ldots, 0, \ldots, n/2 - 1.$$

a) Consider the subspace $\widetilde{\mathcal{H}}_n = \{f \in \mathcal{H}_n : \langle f, e_0 \rangle = 0\}$, which has the kernel $\widetilde{D}_n(x, y) = 2\sum_{k=1}^{n} \cos(k(x - y))$. Show that the matrix $\widetilde{\mathbf{D}}_n := \frac{1}{2n}(\widetilde{D}_n(x_j, x_l))_{j,l=-n/2,\ldots,n/2-1}$ is the identity, that is,

$$\widetilde{D}_n(2\pi j/n, 2\pi l/n) = \begin{cases} 2n & : j = l \\ 0 & : j \neq l. \end{cases}$$

Hint: Use the following fact about geometric series: for $r \neq 1$, we have $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$.

b) Assume to be given observations $(x_k, y_k)_{k=-n/2,\ldots,n/2-1}$. Determine the polynomial $g_n \in \widetilde{H}_n$ which solves the interpolation problem

$$g_n(x_i) = y_i, \quad i = -n/2, \ldots, n/2 - 1.$$

Determine the Fourier coefficients of g_n.

Remark: The Fourier coefficients of g_n are the *discrete Fourier transform* of the vector $(y_k)_{k=-n/2,\ldots,n/2-1}$.

2 Homework

H 1. (Sums of kernel spaces)

Let $k_1, k_2 : \Omega^2 \to \mathbb{R}$ be two positive semi-definite mappings. Consider $k = k_1 + k_2$ given by

$$k(x, y) := k_1(x, y) + k_2(x, y).$$

Show the native space $\mathcal{N}_k$ is given by

$$\mathcal{N}_k = \{f_1 + f_2 \mid f_1 \in \mathcal{N}_{k_1}, f_2 \in \mathcal{N}_{k_2}\}$$

and the norm fufills

$$\|f\|_k^2 = \min\{\|f_1\|_{k_1}^2 + \|f_2\|_{k_2}^2 : f = f_1 + f_2, f_i \in \mathcal{N}_{k_i}\}.$$

Hint: Consider the product space $H = \mathcal{N}_{k_1} \times \mathcal{N}_{k_2}$ with norm given by $\|(f_1, f_2)\|_H^2 = \|f_1\|_{k_1}^2 + \|f_2\|_{k_2}^2$ and find a suitable closed subspace.

(10 Punkte)

H 2. (Kernel smoothing spline)

For even $n \in \mathbb{N}$, let $\mathcal{H}_n$ be the space of trigonometric polynomials as introduced in Sheet 2, G2. Let $x_{-n/2}, \ldots, x_{n/2-1}$ be the sampling points given in G2. Further, let $f : [-\pi, \pi] \to \mathbb{C}$ be a 2π-periodic function. You are given noisy observations

$$y_i = f(x_i) + \epsilon_i, \quad i = -n/2, \ldots, n/2 - 1.$$

The goal is to solve the following kernel smoothing spline optimization problem explicitly:

$$\min_{f_\lambda \in \mathcal{H}_n} \frac{1}{2n} \sum_{i=-n/2}^{n/2} \big(y_i - f_\lambda(x_i)\big)^2 + \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} (f_\lambda^{(m)}(x))^2 dx. \tag{1}$$

Proceed as follows:

a) Determine the trigonometric polynomial $f_{\text{noisy}} \in \mathcal{H}_n$ which solves the interpolation problem

$$f_{\text{noisy}} = y_i, \quad i = -n/2, \ldots, n/2 - 1.$$

b) Use a) to reformulate the optimization problem (1) in terms of the Fourier coefficients of f_{noisy} and f_λ. Determine the Fourier coefficients of the trigonometric polynomial $f_\lambda^* \in \mathcal{H}_n$ which minimizes (1).

c) Compare the Fourier coefficients of f_{noisy} and f_λ^* and interpret the difference between f_{noisy} and f_λ^*.

(10 Punkte)