Wissenschaftliches Rechnen II/Scientific Computing II

Sommersemester 2016
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Exercise sheet 5

To be handed in on **Thursday, 26.05.2016**

G 1. (Regularization)

Let M be a compact, convex metric space and let $R, \rho : M \to [0, \infty)$ be two strictly convex continuous maps. Let $\lambda > 0$.

a) Let us write $R_\lambda(f) = R(f) + \lambda\rho(f)$. Show that

$$\min_{f \in M} R_\lambda(f) \tag{Opt}$$

has a unique minimizer.

b) Show that there is a constant $C_\lambda > 0$ such that solving (Opt) is equivalent to solving

$$\min_{f \in M} R(f) \quad \text{s.t} \quad \rho(f) \le C_\lambda.$$

c) Show that likewise to b) there is a constant $\tilde{C}_\lambda > 0$ such that solving (Opt) is equivalent to solving

$$\min_{f \in M} \rho(f) \quad \text{s.t} \quad R(f) \le \tilde{C}_\lambda.$$

d) Use the insights you gained through a)-c) to explain why regularization typically has a smoothing effect on the solution of the empirical risk minimization problem introduced in the lecture.

Solution.

a) Since M is compact and R_λ is continuous, the minimum in (Opt) exists. Assume there were two different $f, g \in M$ for which the minimum $R_\lambda^* = R_\lambda(f) = R_\lambda(g)$ in (Opt) was attained. Consider a convex combination $h = \alpha f + (1 - \alpha)g$. Then, by strict convexity

$$R_\lambda(h) < \alpha R_\lambda(f) + (1 - \lambda) R_\lambda(g) = R_\lambda^*.$$

This is a contradiction.

b) Let f_λ^* be the minimizer of (Opt). For any $f \in M$ with $R(f) < R(f_\lambda^*)$ we necessarily have $\rho(f) > \rho(f_\lambda^*)$. Hence, for all f in the set $F = \{f \in M : \rho(f) \le \rho(f_\lambda^*) =: C_\lambda\}$ we have $R(f) \ge R(f_\lambda^*)$. Moreover, $f_\lambda^* \in F$. Consequently, f_λ^* is the solution of

$$\min_{f \in M} R(f) \quad \text{s.t} \quad \rho(f) \le C_\lambda.$$

c) The argumentation as in b) with R and ρ switching roles.

□

G 2. (Green's function and kernels)

Consider on the interval $\Omega = [0,1]$ the ordinary differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) = g(x), \quad x \in \Omega.$$

a) Determine the Green's function for the ODE without boundary conditions.

b) Determine the Green's function for the ODE with boundary condition $u(0) = 0$.

The kernels for which spaces have you just recovered?

Solution.

a) Every $G_c(x,y) = \min\{x,y\} + c$ fulfills in the weak, distributional sense

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} G_c(x,y) = \delta(x-y)$$

b) $G(x,y) = \min\{x,y\}$

G_1 is the kernel of $W^1([0,1])$ and G is the kernel of $W_0^1([0,1]) = \{f \in W^1([0,1]) : f(0) = 0\}$. $\square$

G 3. Consider on the interval $\Omega = [0,1]$ for $m \in \mathbb{N}$ the ordinary differential equation

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} u(x) = g(x), \quad x \in \Omega.$$

with boundary condition $u^k(0) = 0$ for $k = 0, \ldots, m-1$. Determine the Green's function $G_m(x,y)$ of the ODE. Show the relation

$$G_{2m}(x,y) = \int_0^1 G_m(x,z) G_m(z,y) \mathrm{d}z.$$

Solution.

$$G_m(x,y) = (x-y)_+^{m-1}/(m-1)!$$

For any function f such that $f^{(k)}(0) = 0$, $k = 0, \ldots, m-1$ and $f^{(m)} \in L_2([0,1])$, we have

$$f(x) = \int_0^1 G_m(x,y) f^{(m)}(y) \mathrm{d}y.$$

This can be verified by interchanging the order of integration in

$$f(x) = \int_0^x \mathrm{d}y_{m-1} \int_0^{y_{m-1}} \mathrm{d}y_{m-2} \ldots \int_0^{y_1} \mathrm{d}y f^{(m)}(y).$$

Since $\frac{\mathrm{d}^m}{\mathrm{d}z^m} G_{2m}(z,y) = G_m(z,y)$ we obtain the desired result. $\square$