Wissenschaftliches Rechnen II/Scientific Computing II

Sommersemester 2016
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Exercise sheet 5

To be handed in on **Thursday, 26.05.2016**

1 Group exercises

G 1. (Regularization)

Let M be a compact, convex metric space and let $R, \rho : M \to [0, \infty)$ be two strictly convex continuous maps. Let $\lambda > 0$.

a) Let us write $R_\lambda(f) = R(f) + \lambda\rho(f)$. Show that

$$\min_{f \in M} R_\lambda(f) \qquad \text{(Opt)}$$

has a unique minimizer.

b) Show that there is a constant $C_\lambda > 0$ such that solving (Opt) is equivalent to solving

$$\min_{f \in M} R(f) \quad \text{s.t} \quad \rho(f) \leq C_\lambda.$$

c) Show that likewise to b) there is a constant $\tilde{C}_\lambda > 0$ such that solving (Opt) is equivalent to solving

$$\min_{f \in M} \rho(f) \quad \text{s.t} \quad R(f) \leq \tilde{C}_\lambda.$$

d) Use the insights you gained through a)-c) to explain why regularization typically has a smoothing effect on the solution of the empirical risk minimization problem introduced in the lecture.

G 2. (Green's function and kernels)

Consider on the interval $\Omega = [0, 1]$ the ordinary differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) = g(x), \quad x \in \Omega.$$

a) Determine the Green's function for the ODE without boundary conditions.

b) Determine the Green's function for the ODE with boundary condition $u(0) = 0$.

The kernels for which spaces have you just recovered?

G 3. Consider on the interval $\Omega = [0, 1]$ for $m \in \mathbb{N}$ the ordinary differential equation

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} u(x) = g(x), \quad x \in \Omega.$$

with boundary condition $u^k(0) = 0$ for $k = 0, \ldots, m - 1$. Determine the Green's function $G_m(x, y)$ of the ODE. Show the relation

$$G_{2m}(x, y) = \int_0^1 G_m(x, z) G_m(z, y) \mathrm{d}z.$$

2 Homework

H 1. (Semiparametric representer theorem)

Suppose that in addition to the assumptions of Theorem 41 in the lecture we are given a set of M real-valued functions $(\psi_j)_{j=1}^M$, each mapping from Ω to $\mathbb{R}$, which have the property that the $m \times M$-matrix $(\psi_j(x_i))_{\substack{i=1,\dots,N\\ j=1,\dots,M}}$ has rank M. Prove the following statement:

Any function $\tilde{f} = f + h$, with $f \in \mathcal{H}$ and $h \in \text{span}\{\psi_1, \dots, \psi_M\}$, which minimizes the regularized risk

$$R_{\text{reg},\ell}(\tilde{f}) = \frac{1}{N}\sum_{i=1}^{N} \ell(x_i, y_i, \tilde{f}(x_i)) + \lambda s(\|f\|_{\mathcal{H}}), \quad \lambda > 0,$$

admits a representation $\tilde{f}(x) = \sum_{i=1}^N \alpha_i k(x_i, x) + \sum_{j=1}^M \beta_j \psi_j(x)$ with $\alpha_i, \beta_j \in \mathbb{R}$.

Hint: start with a decomposition of $\tilde{f}$ into a parametric part, a kernel part, and an orthogonal contribution and evaluate the loss and regularization terms independently.

(4 Punkte)

H 2. (Sobolev space)

Let $\phi_k(x) = x^{k-1}/(k-1)!$ for $k \in \mathbb{N}$. Show that the Sobolev space

$$W^m([0,1]) := \left\{ f : f, f', \dots, f^{m-1} \text{ absolutely continuous }, f^{(m)} \in L_2([0,1]) \right\}$$

endowed with the inner product

$$\langle f, g \rangle_{W^m} := \sum_{k=0}^{m-1} [\frac{\mathrm{d}^k}{\mathrm{d}x^k} f](0) [\frac{\mathrm{d}^k}{\mathrm{d}x^k} g](0) + \int_0^1 f^{(m)}(x) g^{(m)}(x) \mathrm{d}x$$

has the reproducing kernel $R(x,y) = \sum_{k=1}^m \phi_k(x)\phi_k(y) + \int_0^1 G_m(y,z) G_m(x,z) \mathrm{d}z$, where G_m is the Green's function computed in G3.

Hint: start with the Taylor expansion

$$f(x) = \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0) + \int_0^1 \frac{(x-u)_+^{m-1}}{(m-1)!} f^{(m)}(u) \mathrm{d}u$$

and write the Sobolev space as a sum of two orthogonal spaces, for which you determine the kernels first.

(6 Punkte)

H 3. (Green's function)

Consider on the interval $\Omega = [0,1]$ the ordinary differential equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) = g(x), \quad x \in \Omega.$$

with boundary conditions $u(0) = u(1) = 0$. Determine the Green's function $G(x,y)$. You can assume to know the following about the Green's function $G(x,y)$:

- G is continuous along the diagonal $x = y$,
- for any fixed $y \in (0,1)$, $G'(\cdot, y)$ has a jump discontinuity at $x = y$ of the form

$$\lim_{x \to y^-} G'(x,y) = 1 + \lim_{x \to y^+} G'(x,y).$$

(4 Punkte)

H 4. (Regularized least-squares regression)

This is a programming exercise. As usual, you find the tasks in the accompanying notebook on the lecture's website.

(6 Punkte)