# V4E2 - Numerical Simulation

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## Exercise sheet 1.

To be handed in on **Thursday, 27.4.2017**.

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Hamiltonian and $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the problem

$$H(x, u, Du) = 0, \quad \forall x \in \Omega.$$

The definition of **viscosity solution** for this problem was given in the lecture.

**Exercise 1.** Check that

$$u(x) = \begin{cases} x & , \quad 0 < x \leq \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} < x < 1 \end{cases}$$

is a viscosity solution of $H(x, u, Du) := |u'(x)| - 1 = 0$, $x \in (0, 1)$. Is u a viscosity solution of $-|u'(x)| + 1 = 0$ in $(0, 1)$?

(4 Punkte)

**Exercise 2.** Prove: Let $v \in C(\Omega)$ and suppose that $x_0 \in \Omega$ is a strict maximum point for $v$ in $\overline{B}(x_0, \delta) \subset \Omega$. If $v_n \in C(\Omega)$ converges locally uniformly to $v$ in $\Omega$, then there exists a sequence $\{x_n\}$ such that

$$x_n \to x_0, \quad v_n(x_n) \geq v_n(x) \quad \forall x \in \overline{B}(x_0, \delta).$$

(4 Punkte)

An alternative way defining viscosity solutions is provided with the help of sub- and super-differentials. In the first exercise we will give some details on that issue.

**Definition 1.** Let $\Omega$ be an open set in $\mathbb{R}^d$ and $v : \Omega \to \mathbb{R}$. The super-differential $D^+v(x)$ of $v$ at $x \in \Omega$, is defined as the set

$$D^+v(x) := \Big\{ p \in \mathbb{R}^d : \limsup_{\substack{y \to x \\ y \in \Omega}} \frac{v(y) - v(x) - p \cdot (y - x)}{|y - x|} \leq 0 \Big\}.$$

The sub-differential $D^-v(x)$ of $v$ at $x \in \Omega$, is defined as the set:

$$D^-v(x) := \Big\{ q \in \mathbb{R}^d : \liminf_{\substack{y \to x \\ y \in \Omega}} \frac{v(y) - v(x) - q \cdot (y - x)}{|y - x|} \geq 0 \Big\}.$$

**Exercise 3.** a) Let

$$v_1(x) := |x|.$$

Compute $D^+v_1(0)$ and $D^-v_1(0)$.

b) Let

$$v_2(x) := \begin{cases} 0 & , \quad x \leq 0 \\ \frac{1}{2}bx^2 + ax & , \quad x > 0. \end{cases}$$

Compute $D^+v_2(0)$.

(5 Punkte)

**Exercise 4.** Prove: If $u : \mathbb{R}^d \to \mathbb{R}$ is convex (i.e. $u(\lambda x + (1-\lambda)y) \le \lambda u(x) + (1-\lambda)u(y)$, for any $x, y \in \mathbb{R}$, $\lambda \in [0,1]$), then its sub-differential at $x$ in the sense of convex analysis is the set

$$\partial_c u(x) := \{p \in \mathbb{R}^d : u(y) \ge u(x) + p \cdot (y-x),\ \forall y \in \mathbb{R}^d\}.$$

Show that if $u$ is convex then $\partial_c u(x) = D^- u(x)$.

(4 Punkte)