V4E2 - Numerical Simulation

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Exercise sheet 10.

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The infinite horizon problem

Let y_x denote the unique solution of the Cauchy problem

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) \\ y(0) = x. \end{cases}$$

We aim to minimize the cost

$$J(x, \alpha) := \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt.$$

For that purpose we define the value function as

$$v(x) := \inf_{\alpha \in \mathcal{A}} J(x, \alpha).$$

Prerequisites

Let $A \subset \mathbb{R}^M$ compact.

(A_0)

$$\begin{cases} A \text{ is a topological space,} \\ f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ is continuous,} \end{cases}$$

(A_1) f is bounded on $B(0, R) \times A$ for all $R > 0$,

(A_2) there is a modulus ω_f such that

$$|f(y, a) - f(x, a)| \leq \omega_f(|x - y|, R),$$

for all $x, y \in B(0, R)$ and $R > 0$.

(A_3)

$$(f(x, a) - f(y, a)) \cdot (x - y) \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R}^N, a \in A.$$

(A_4)
- ℓ is continuous,
- there are modulus ω_ℓ and a constant M such that

$$|\ell(x, a) - \ell(y, a)| \leq \omega_\ell(|x - y|)$$

  and

$$|\ell(x, a)| \leq M,$$

  for all $x, y \in \mathbb{R}^N$ and $a \in A$,
- $\lambda > 0$

Exercises

Exercise 1. (Variable interest rate)

Let $\lambda : \mathbb{R}^N \times A \to \mathbb{R}$ satisfy $0 < \lambda_0 \le \lambda(x,a) \le M'$ and $|\lambda(x,a) - \lambda(y,a)| \le \omega_\lambda(|x-y|)$, where ω_λ is a modulus, for all $x, y \in \mathbb{R}^N$ and $a \in A$. Consider the payoff

$$J(x,a) := \int_0^\infty \exp\Big(-\int_0^t \lambda(y_x(s), \alpha(s))ds\Big)\ell(y_x(t), \alpha(t))dt$$

under the hypotheses $(A_0) - (A_4)$.

(i) Prove that the value function $v = \inf_\alpha J$ satisfies the following DPP:

$$v(x) = \inf_{\alpha \in \mathcal{A}} \Big\{ \int_0^t \ell(y_x(s), \alpha(s)) \exp\Big(-\int_0^s \lambda(y_x(\tau), \alpha(\tau))d\tau\Big) + v(y_x(t)) \exp\Big(-\int_0^t \lambda(y_x(\tau), \alpha(\tau))d\tau\Big)\Big\}.$$

(ii) Prove that v is a viscosity solution of

$$\sup_{a \in A}\{\lambda(x,a)v - f(x,a)\cdot Dv - \ell(x,a)\} = 0, \quad x \in \mathbb{R}^N.$$

(6 Punkte)

Convergence of more general approximation schemes

We denote upper semicontinuous and lower semicontinuous envelopes of a real valued function u by

$$u^*(x) = \limsup_{y \to x} u(y) \quad \text{and} \quad u_*(x) = \liminf_{y \to x} u(y).$$

Definition 1.

- u is a viscosity sub-solution of $H(x,u,Du) = 0$ in Ω if for all functions $\varphi \in C^1(\Omega)$, for all $x \in \Omega$, local maximum of $u^* - \varphi$ such that $u^*(x) = \varphi(x)$, we have:

$$H_*(x, \varphi(x), D\varphi(x)) \le 0$$

- u is a viscosity super-solution of $H(x,u,Du) = 0$ in Ω if for all functions $\varphi \in C^1(\Omega)$, for all $x \in \Omega$, local minimum of $u_* - \varphi$ such that $u_*(x) = \varphi(x)$, we have:

$$H^*(x, \varphi(x), D\varphi(x)) \ge 0$$

As in the lecture we assume Ω as a bounded polyhedral domain. Let

$$\Omega_h := h\mathbb{Z}^d \cap \Omega$$

be a space discretization with parameter h. We are interested in the HJB equation

$$H(x,u,Du) = 0, \quad \forall x \in \Omega$$

with

$$u(x) \le b(x), \quad \forall x \in \partial\Omega$$

associated to the Hamiltonian

$$H(x,u,a) := \sup_{a \in A}(-f(x,a)\cdot Du(x) - \ell(x,a)).$$

It is well known that under certain conditions a unique viscosity solution of the equation above is provided by the value function v.

Let S_h be an operator on the space of bounded functions on Ω_h. We are concerned with the convergence of the solution v_h to the dynamic progamming equation:

$$v_h(x_i) = S_h[v_h](x_i), \quad \text{for } x_i \in \Omega_h \tag{1}$$

with the boundary condition

$$v_h(x_i) \leq b(x_i) \quad \text{for } x_i \in \partial\Omega_h \tag{2}$$

Exercise 2. Assume Lipschitz continuity of f in the space variable. Furthermore assume the corresponding value function as continuous in Ω. We make the following additional assumptions on S_h :

(i) Monotonicity: if $v_1 \leq v_2$ then $S_h[v_1] \leq S_h[v_2]$

(ii) For any constant c (approximately invariant w.r.t. addition of constants),

$$S_h[v + c] = S_h[v] + c(1 + O(h))$$

(iii) Consistency in the form of:

$$\lim_{(x_i)_h \xrightarrow{h\to 0} x} \frac{1}{h}[v - S_h[v]]((x_i)_h) = H(x, v(x), Dv(x)).$$

Prove that S_h is a convergent approximation scheme, i.e. the solutions v_h of (1) and (2) satisfy

$$\lim_{(x_i)_h \xrightarrow{h\to 0} x} v_h((x_i)_h) = v(x)$$

uniformly.
Hints: Define the largest and smallest limit functions

$$v_{\text{sup}} := \limsup_{\xi \xrightarrow{h\to 0} x} v_h(\xi)$$

and

$$v_{\text{inf}} := \liminf_{\xi \xrightarrow{h\to 0} x} v_h(\xi).$$

Prove that they are respectively sub- and super viscosity solutions. W.l.o.g. assume the extrema to be a global one. Finally use the following comparison result:

Lemma 1. Under the assumptions above the HJB equation has a weak comparison principle, i.e. for any viscosity sub-solution $\underline{u}$ and super-solution $\overline{u}$ and for all $x \in \Omega$ we have:

$$\underline{u}(x) \leq \overline{u}(x).$$