# V4E2 - Numerical Simulation

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Prof. Dr. J. Garcke
G. Byrenheid

universitätbonn

## Exercise sheet 2.

To be handed in on **Thursday, 04.05.2017**.

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Hamiltonian and $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the problem

$$H(x, u, Du) = 0, \quad \forall x \in \Omega. \tag{1}$$

We assume

[A1] $H(\cdot, \cdot, \cdot)$ is uniformly continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^d$

[A2] $H(x, u, \cdot)$ is convex on $\mathbb{R}^d$

[A3] $H(x, \cdot, p)$ is monotone on $\mathbb{R}$.

**Exercise 1.** Show by a density argument that an equivalent definition of viscosity solution for (1) can be given by using $C^\infty(\Omega)$ instead of $C^1(\Omega)$ as the 'test function space'. (Hint: Friedrichs mollifier)

(4 Punkte)

**Exercise 2.** Show by exhibiting an example that is false in general that if $u$, $v$ are viscosity solutions of (1) the same is true for $u \wedge v$, $u \vee v$.

(4 Punkte)

**Exercise 3.** Suppose that the equation $H_n(x, u_n(x), Du_n(x)) = 0$ has a classical solution $u_n \in C^1(\Omega)$ for $n = 1, 2, \ldots$. Show that, under the assumptions of Proposition 6, $u = \lim_{n\to\infty} u_n$ is a viscosity solution of

$$-H(x, u(x), Du(x)) = 0.$$

(4 Punkte)

(*) Solve **one** of the following exercises. By solving the second one you can earn **extra points**.

**Exercise 4.** Let $H(x, p) = \sup_{\alpha\in A}\{-f(x, a) \cdot p - \ell(x, a)\}$, with $A$ compact, $f$ and $\ell$ continuous. Assume also that, for all $x, y$

$$|f(x, a) - f(y, a)| \leq L|x - y|, \quad |\ell(x, a) - \ell(y, a)| \leq \omega(|x - y|)$$

where the constant $L$ and the modulus $\omega$ are independent of $a \in A$. Show that $H$ satisfies

$$|H(x, p) - H(y, p)| \leq \omega_1(|x - y|(1 + |p|)).$$

($\omega_1 : [0, +\infty[ \to [0, +\infty[$ is continuous nondecreasing with $\omega_1(0) = 0$).

(4* Punkte)

**Exercise 5.** Take $H(x,p) = \sup_{a\in A}\{-f(x,a)\cdot p - \ell(x,a)\}$ with $f$ continuous on $\mathbb{R}^N \times A$. Assume also that

$$\exists r > 0 : B(0,r) \subseteq \overline{co} f(x,A), \quad \forall x \in \mathbb{R}^N$$

holds. Show that

$$\sup_{a\in A}\{-f(x,a)\cdot p\} = \sup_{\xi\in\overline{co} f(x,A)}\{-\xi\cdot p\} \geq r|p|.$$

Prove than that $H$ satisfies the coercivity condition

$$H(x,p) \to +\infty \quad \text{as} \quad |p| \to +\infty$$

provided $\ell$ is bounded.

(4* Punkte)