V4E2 - Numerical Simulation

Sommersemester 2017
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Exercise sheet 3.

To be handed in on **Thursday, 11.05.2017**.

Exercise 1. Prove: The result of Theorem 7 remains true if assumption (A4) is replaced by the coercivity condition

$$r + H(x,p) \to +\infty \quad \text{as} \quad |p| \to +\infty,$$

uniformly with respect to $x \in \overline{\Omega}$, $|r| < R$, $(\forall R > 0)$. **Update:** Assume additionally $H \in C(\overline{\Omega} \times \mathbb{R}^d)$. Hint: Modify the end of the proof of Theorem 7. Observe that the inequality

$$u_1(x_\varepsilon) + H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big) \leq 0$$

in the proof of Theorem 7 and coercivity imply that $|(x_\varepsilon - y_\varepsilon)/\varepsilon|$ is uniformly bounded as $\varepsilon \to 0^+$.

(4 Punkte)

Let $L : \mathbb{R}^n \to \mathbb{R}$ be a Lagrangian with

(i) $v \to L(v)$ is convex,

(ii) $\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$.

We define $u(x,t)$ by the Hopf-Lax formula as

$$u(x,t) := \min_{y\in\mathbb{R}^n} \Big\{ tL\Big(\frac{x-y}{t}\Big) + g(y) \Big\}.$$

Exercise 2. Prove: For each $x \in \mathbb{R}^n$ and $0 \leq s < t$, we have

$$u(x,t) = \min_{y\in\mathbb{R}^n} \Big\{ (t-s)L\Big(\frac{x-y}{t-s}\Big) + u(y,s)\Big)\Big\}.$$

Hints for $\leq$:

- Fix $z \in \mathbb{R}^n$ such that:
$$u(y,s) = sL\Big(\frac{x-z}{s}\Big) + g(z),$$
- use convexity with
$$\frac{x-z}{t} = \Big(1 - \frac{s}{t}\Big)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s},$$
- Use/assume the continuity of u.

(6 Punkte)

We define the Legendre transform of L as

$$L^*(p) := \sup_{v\in\mathbb{R}^n} \{p \cdot v - L(v)\}, \ (p \in \mathbb{R}^n).$$

The corresponding Hamiltonian is given by

$$H := L^*.$$

Exercise 3. Prove:

a) The mapping $p \to H(p)$ is convex,

b) it fulfills the coercivity condition

$$\lim_{|v|\to\infty} \frac{H(v)}{|v|} = +\infty,$$

c) $L = H^*$.

Hints for c):

- $L \geq H^*$: elementary,
- $L \leq H^*$:

  $$H^*(v) = \sup_{p\in\mathbb{R}^n} \{p \cdot v - \sup_{r\in\mathbb{R}^n} \{p \cdot r - L(r)\}\}$$

  convexity of L implies

  $$\exists s \in \mathbb{R}^n : \quad L(r) \geq L(v) + s \cdot (r - v).$$

(6 Punkte)