Exercise sheet 4.

To be handed in on **Thursday, 18.05.2017**.

Exercise 1. Prove:

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. $0 \leq t \leq T$ the differential inequality

$$\eta'(t) \leq \omega(t)\eta(t) + \psi(t)$$

where $\omega(t)$ and $\psi(t)$ are nonnegative, integrable functions on $[0,T]$. Then

$$\eta(t) \leq e^{\int_0^t \omega(s)ds}\Big[\eta(0) + \int_0^t \psi(s)ds\Big]$$

for all $0 \leq t \leq T$.

(ii) Let $\phi(\cdot)$ be a nonnegative, integrable function on $[0,T]$ which satisfies for a.e. $0 \leq t \leq T$ the integral inequality

$$\phi(t) \leq C_2 + \int_0^t C_1\phi(s)ds$$

for constants $C_1, C_2 > 0$. Then

$$\phi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$. Hints: (i): consider $\frac{d}{ds}(\eta(s)e^{-\int_0^s \omega(r)dr})$ (ii): Use (i) to prove (ii).

(6 Punkte)

Exercise 2. Prove: Assuming

(i) $f : \mathbb{R}^n \times A \to \mathbb{R}^n$ is continuous such that

$$|f(x,\alpha)| \leq C, \quad |f(x,\alpha) - f(y,\alpha)| \leq C|x-y|, \quad \forall x,y \in \mathbb{R}^n, \alpha \in A$$

(ii) $\psi : \mathbb{R}^n \to \mathbb{R}$

$$|\psi(x)| \leq C, \quad |\psi(x) - \psi(y)| \leq C|x-y|, \quad \forall x,y \in \mathbb{R}^n$$

(iii) $\ell : \mathbb{R}^n \times A \to \mathbb{R}$

$$|\ell(x,\alpha)| \leq C, \quad |\ell(x,\alpha) - \ell(y,\alpha)| \leq C|x-y|, \quad \forall x,y \in \mathbb{R}^n, \alpha \in A.$$

Then

$$V(x,t) := \inf_{\alpha \in \mathcal{A}} J_{x,t}(\alpha) = \inf_{\alpha \in \mathcal{A}} \int_t^T \ell(y_{x,t}(s), \alpha(s))dt + \psi(y_{x,t}(T)) \tag{1}$$

is bounded and Lipschitz continuous, i.e.

$$|V(y,s)| \leq C', \quad |V(y,s) - V(y',s')| \leq C'(|s-s'| + |y-y'|), \quad \forall x,y \in \mathbb{R}^d, 0 \leq s,s' \leq T$$

(As usual: $y_{x,t}(s)$ is the unique solution of the state dynamics generated by x, t and f) Hints:

- consider 2 trajectories $z(s) := y_{x,t}(s, \alpha_\varepsilon)$, $z'(s) := y_{x',t'}(s, \alpha_\varepsilon)$ with different initial data and where α_ε is an almost optimal control for the initial data (x, t),
- prove $|z'(t) - z(t)| \leq C|t - t'| + |x' - x|$,
- prove $|z'(s) - z(s)| \leq e^{CT}(C|t - t'| + |x' - x|)$ using Exercise 1),
- estimate $J_{x,t}(\alpha_\varepsilon) - J_{x',t'}(\alpha_\varepsilon)$.

(6 Punkte)

Exercise 3. Given the initial data $y(t_0) = x_0$. We define the function

$$h(t) := \int_{t_0}^{t} \ell(y_{x_0,t_0}(s), \alpha(s))ds + V(y_{x_0,t_0}(t), t).$$

where V is the value function known from (1). Prove:

(i) h is nondecreasing for any control α,

(ii) h is constant if and only if the control α is optimal.

(4 Punkte)