V4E2 - Numerical Simulation

Sommersemester 2017
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Exercise sheet 5.

To be handed in on **Tuesday, 30.05.2017**.

Definition 1. The scheme S (with a uniform infinite space grid) is said to be invariant by translation if, defined the translation operator Θ_i such that

$$(\Theta_i V)_j = V_{j+e_i},$$

we have for any $i = 1, \dots, d$

$$S(\Delta, \Theta_i V) = \Theta_i S(\Delta, V)$$

Exercise 1. We use the notation

$$D_{i,j}[V] = \frac{v_{j+e_i} - v_j}{\Delta x_i}.$$

Prove that for a monotone, invariant scheme which is conserving constants the inequality

$$\frac{\|D_i[S(V)]\|_\infty}{\|D_i[V]\|_\infty} \leq 1$$

holds. This yields Lipschitz stability for the scheme.

(3 Punkte)

We study finite difference schemes now. Let us describe different kind of schemes by considering the one-way wave equation

$$u_t + a u_x = 0, \tag{1}$$

where a is a constant, t represemts the time and x the spatial variable. The idea of difference schemes is it to replace derivatives by differences. Using the notation

$$u_m^n := u(t_n, x_m)$$

we obtain for (1) the following discretizations (examples)

(i) forward-time forward-space scheme

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + a \frac{v_{m+1}^n - v_m^n}{\Delta x} = 0$$

(ii) forward-time backwars-space scheme

$$\frac{v_m^{n+1} - v_m^n}{\Delta t} + a \frac{v_m^n - v_{m-1}^n}{\Delta x} = 0$$

(iii) Lax-Friedrichs

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{\Delta t} + a \frac{v_{m+1}^n - v_{m-1}^n}{2\Delta x} = 0.$$

Exercise 2. Solve the system using a computer and an environment of your choice, for example Python/Numpy

$$u_t + \frac{1}{3}(t-2)u_x + \frac{2}{3}(t+1)\omega_x + \frac{1}{3}u = 0,$$
$$\omega_t + \frac{1}{3}(t+1)u_x + \frac{1}{3}(2t-1)\omega_x - \frac{1}{3}\omega = 0$$

by the Lax-Friedrichs scheme: i.e., each time derivative is approximated as it is for the scaler equation and the spatial derivatives are approximated by central differences. The initial values are

$$u(0,x) = \max(0, 1-|x|),$$
$$\omega(0,x) = \max(0, 1-2|x|).$$

Consider values of x in $[-3,3]$ and t in $[0,2]$. Take $\Delta x = \frac{1}{20}$ and $\Delta t = \frac{1}{40}$. At each boundary set $u = 0$, and set ω equal to the newly computed value one grid point in from the boundary. Describe the solution behavior for t in the range $[1.5, 2]$. Show some plots.

(10 Punkte)

For reasons of notation we use for the rest of the exercise sheet a simplified notation of scheme. Let S_Δ be a finite difference operator acting on a solution u of a partial differential equation $Pu = f$ using discretization parameters $\Delta = (\Delta t, \Delta x)$. We denote S_Δ as a scheme.

Definition 2. Given a partial differential equation $Pu = f$, and a finite difference scheme $S_\Delta u = f$, we say that the finite difference scheme S_Δ is consistent with the partial differential equation if for any smooth function $\phi(t,x)$

$$P\phi - S_\Delta\phi \to 0 \quad \text{as} \quad \Delta t, \Delta x \to 0.$$

The convergence being pointwise convergence at each point (t,x).

Exercise 3. Prove consistency of the Lax-Friedrichs scheme

$$S_\Delta\phi = \frac{\phi^{n+1}_m - \frac{1}{2}(\phi^n_{m+1} + \phi^n_{m-1})}{\Delta t} + a\frac{\phi^n_{m+1} - \phi^n_{m-1}}{2\Delta x}$$

with $|\Delta t|^{-1}|\Delta x|^2 \to 0$ as $\Delta t, \Delta x \to 0$.

(4 Punkte)

Definition 3. A finite difference scheme $S_\Delta v = 0$ for a first order equation is stable in a stability region Λ (describes the possible relations of $\Delta t, \Delta x$ tending to zero) if there is an integer J such that for any positive time T, there is a constant C_T such that

$$\Delta x \sum_{m=-\infty}^{\infty} |v^n_m|^2 \leq C_T \Delta x \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} |v^j_m|^2$$

for $0 \leq n\Delta t \leq T$, with $(\Delta t, \Delta k) \in \Lambda$.

Exercise 4. Show that schemes of the form

$$v^{n+1}_m = \alpha v^n_{m+1} + \beta v^n_{m-1}$$

are stable if

$$|\alpha| + |\beta| \leq 1.$$

Conclude that the Lax-Friedrichs scheme is stable if $|a\frac{\Delta t}{\Delta x}| \leq 1$.

(4 Punkte)