V4E2 - Numerical Simulation

Sommersemester 2017
Prof. Dr. J. Garcke
G. Byrenheid

Exercise sheet 6.

To be handed in on **Tuesday,13.06.2017**.

We consider the linear advection equation in one space dimension with constant coefficient

$$u_t(x,t) + cu_x(x,t) = 0, \quad (x,t) \in \mathbb{R} \times [0,T]$$

$$u(x,0) = u_0(x).$$

Exercise 1. (i) Prove consistency for the forward in time - forward in space scheme

$$0 = \frac{v_i^{j+1} - v_i^j}{\Delta t} + c\frac{v_{i+1}^j - v_i^j}{\Delta x}.$$

(ii) We set $c = 1$. Consider the initial condition

$$u_0(x) = \begin{cases} 1 & -1 \leq x \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Where we use as initial condition for S_Δ

$$v_i^0 = \begin{cases} 1 & -1 \leq i\Delta x \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

What can you observe concering convergence of the scheme for the equation stated above? An analytic solution is provided by $u(x,t) = u_0(x - ct)$.

(6 Punkte)

Exercise 2. Assume $u_0 \in C^1(\mathbb{R})$. Find a solution using the methods of characteristics. Implement (using a computer and an environment of your choice, for example Python/Numpy) the forward in time - centered in space (FTCS) scheme

$$v_i^{j+1} = v_i^j - \frac{c\Delta t}{2\Delta x}(v_{i+1}^j - v_{i-1}^j),$$

the Upwind scheme

$$v_i^{j+1} = v_i^j - \frac{c\Delta t}{\Delta x}(v_i^j - v_{i-1}^j)$$

(if $c > 0$) and the Lax-Scheme

$$v_i^{j+1} = \frac{1}{2}(v_{i+1}^j + v_{i-1}^j) - \frac{c\Delta t}{2\Delta x}(v_{i+1}^j - v_{i-1}^j)$$

with the parameters

- $\Omega = [0, 10]$,
- $u_0(x) = e^{-10(x-2)^2}$,
- $\Delta x = \Delta t = 0.05$,

- $c = 0.5$.

Plot the approximated solutions for time $t = 0,\ 50,\ 100,\ 150,\ 200$. Play with the parameters Δx, Δt and the velocity c such that the so called CFL condition is not fulfilled anymore, i.e. $\frac{c\Delta t}{\Delta x} > 1$. Is there something interesting to observe comparing the plots?

(6 Punkte)

Definition 1. Let $(\Delta x_m, \Delta t_m)$ and (x_{j_m}, t_{k_m}) be generic sequences, i.e. for $m \to \infty$

$$(\Delta x_m, \Delta t_m) \to 0 \quad \text{and} \quad (x_{j_m}, t_{n_m}) \to (x, t).$$

Then the scheme S is said to be monotone (in the generalized sense) if it satisfies the following conditions

- $v_{j_m} \leq \phi_{j_m}$, then $S_{j_m}(\Delta_m, V) \leq S_{j_m}(\Delta_m, \Phi) + o(\Delta t_m)$,
- $\phi_{j_m} \leq v_{j_m}$, then $S_{j_m}(\Delta_m, \Phi) \leq S_{j_m}(\Delta_m, V) + o(\Delta t_m)$

for any smooth function $\phi(x)$.

Exercise 3. Prove Theorem 29 (Barles and Souganidis convergence theorem) where the monotonicity assumption is replaced by the weaker generalized monotonicity assumption. Assume additionally a relaxed monotonicity property for the interpolation operator

- if $v_j \leq \phi_j$ for any j such that $x_j \in st(x)$, then $I[V](x) \leq I[\Phi](x) + o(\Delta t)$,
- if $\phi_j \leq v_j$ for any j such that $x_j \in st(x)$, then $I[\Phi](x) \leq I[V](x) + o(\Delta t)$.

(6 Punkte)

Die Fachschaft Mathematik feiert am 1.6. ihre Matheparty in der N8schicht. Der VVK findet am Mo. 29.05., Di. 30.05. und Mi 31.05. in der Mensa Poppelsdorf statt. Alle weitere Infos auch auf fsmath.uni-bonn.de