Exercise sheet 7.

To be handed in on **Tuesday, 20.06.2017**.

We consider the linear advection equation in one space dimension with constant coefficient $c > 0$

$$u_t(x,t) + cu_x(x,t) = 0, \quad u(x,0) = u_0(x). \tag{1}$$

The exact solution of the discretized equations (finite differences) satisfies a PDE which is generally different from the one to be solved

Original equation		Modified equation solved by $u^{n+1} = S(\Delta, u^n)$
$\frac{\partial u}{\partial t} + \mathcal{L}u = 0$	$\sim$	$\frac{\partial u}{\partial t} + \mathcal{L}u = \sum_{p=1}^{\infty} \alpha_{2p} \frac{\partial^{2p} u}{\partial x^{2p}} + \sum_{p=1}^{\infty} \alpha_{2p+1} \frac{\partial^{2p+1} u}{\partial x^{2p+1}}$

Exercise 1. (i) Prove that the numerical solution of (1) by the upwind scheme

$$\frac{u_j^{i+1} - u_j^i}{\Delta t} + c\frac{u_j^i - u_{j-1}^i}{\Delta x} = 0$$

corresponds to a solution of

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\lambda)\frac{\partial^2 u}{\partial x^2} + \frac{c(\Delta x)^2}{6}(3\lambda - 2\lambda^2 - 1)\frac{\partial^3 u}{\partial x^3} + \ldots$$

where $\lambda = \frac{c\Delta t}{\Delta x}$ and $\ldots$ contains derivatives of order > 3. Hints:

- Expand all nodal values in the difference scheme in a double Taylor series about a single point (x_i, t_j) of the space-time mesh to obtain a PDE
- Express high-order time derivatives as well as mixed derivatives in terms of space derivatives using this PDE to transform it into the desired form

(ii) We set $c = 1$. Consider the initial condition

$$u_0(x) = \begin{cases} 1 & -1 \le x \le 0, \\ 0 & \text{elsewhere,} \end{cases}$$

where we use for the difference scheme

$$u_i^0 = \begin{cases} 1 & -1 \le i\Delta x \le 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Implement the upwind scheme for suitable parameters $0 < \lambda = \frac{\Delta t}{\Delta x} \le 1$ in a suitable domain (cf. analytic solution). At the left boundary set the newly computed $u_{j^*}^{i+1} = u_{j^*+1}^{i+1}$ $(= 0)$. As known from Sheet 6, Ex. 1 an analytic solution is given by $u(x,t) = u_0(x - ct)$. What qualitative effects do you observe comparing numerical and analytic solution over time?

(iii) Compare this qualitative effects with that which appear using the Lax-Friedrichs scheme

$$\frac{u_j^{i+1} - u_j^i}{\Delta t} + \frac{u_{j+1}^i - u_{j-1}^i}{2\Delta x} = 0.$$

For a sufficient large domain assume similar to (ii) $u^{i+1} = 0$ for the left and right boundary.

(10 Punkte)

Exercise 2. In the lecture the upwind scheme $S(\Delta, V)$ for the convex Hamilton-Jacobi equation was introduced. Prove monotonicity of the upwind scheme by showing $\frac{\partial}{\partial v_i} S_j(\Delta, V) \geq 0$.

(6 Punkte)

Exercise 3. On Sheet 6, Exercise 1 a misprint appeared in the representation of the scheme. The correct scheme has the presentation

$$0 = \frac{v_i^{j+1} - v_i^j}{\Delta t} + c\frac{v_{i+1}^j - v_i^j}{\Delta x}.$$

Since this misprint caused trouble for some students you can reinsert Exercise 1 together with this sheet. If you insert a new solution then only this will be rated.

In general: if questions appear, do not hestiate to ask the tutor or contact glenn.byrenheid@hcm.uni-bonn.de.