# V4E2 - Numerical Simulation

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## Exercise sheet 8.

To be handed in on **Tuesday, 27.06.2017**.

## The minimum time problem

### Description of the problem

We study problems with inital state $x \in \mathcal{T}^c := \mathbb{R}^N \setminus \mathcal{T}$, whose dynamics

$$\begin{cases} y'(t) = f(y(t), \alpha(t)), \ t > 0, \\ y(0) = x, \end{cases}$$

is stopped and the payoff computed when the system reaches the closed set $\mathcal{T}$, where $\operatorname{int} \mathcal{T} \neq \emptyset$, $\partial\mathcal{T}$ is sufficiently regular. We are interested in the minimal time function

$$T(x) := \inf_{\alpha \in \mathcal{A}} t_x(\alpha)$$

where the first time of arrival is defined by

$$t_x(\alpha) := \begin{cases} +\infty & \text{if } \{t : y_x(t, \alpha) \in \mathcal{T}\} = \emptyset, \\ \inf\{t : y_x(t, \alpha) \in \mathcal{T}\} & \text{otherwise.} \end{cases}$$

Additionally we define the reachable set as

$$\mathcal{R} := \{x \in \mathbb{R}^N : T(x) < +\infty\},$$

which describes the set of initial states from which it is possible to reach the target $\mathcal{T}$.

### Further prerequisites

Let $A \subset \mathbb{R}^M$ compact.

($\mathrm{A}_0$)

$$f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ is continuous,}$$

($\mathrm{A}_3$)

$$(f(x, a) - f(y, a)) \cdot (x - y) \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R}^N, a \in A.$$

### Exercises

**Exercise 1.** (Dynamic Programming Principle)

Prove: Assume $(A_0)$, $(A_3)$. Then, for all $x \in \mathcal{R}$,

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{\min\{t, t_x(\alpha)\} + \chi_{\{t \leq t_x(\alpha)\}} T(y_x(t, \alpha))\},$$

for all $t \geq 0$ and

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{t + T(y_x(t, \alpha))\},$$

for all $t \in [0, T(x)]$.

(6 Punkte)

**Exercise 2.** Let a function $S : \mathbb{R}^N \to [0, +\infty]$ satisfy the Dynamic Programming Principle (DPP) at given point $x \in \mathcal{R}$, that is

$$S(x) = \inf_{\alpha \in \mathcal{A}} \{t \wedge t_x(\alpha) + \chi_{\{t \leq t_x(\alpha)\}} S(y_x(t, \alpha))\}, \quad \text{for all } t \geq 0.$$

Prove that $S(x) = T(x)$.

(6 Punkte)

**Exercise 3.** Prove: If $\mathcal{R} \backslash \mathcal{T}$ is open and $T$ is continuous, then $T$ is is a viscosity solution of

$$\begin{cases} H(x, DT(x)) = 0, & x \in \mathcal{R} \backslash \mathcal{T}, \\ T(x) = 0, & x \in \partial \mathcal{T}, \end{cases}$$

where

$$H(x, p) := \sup_{\alpha \in A} \{-p \cdot f(x, a)\} - 1.$$

Hints:

- Ignore/skip the boundary condition.
- Apply the DPP.

(6 Punkte)