V4E2 - Numerical Simulation

Sommersemester 2017
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Exercise sheet 9.

To be handed in on **Tuesday, 04.07.2017**.

The infinite horizon problem

Let y_x denote the unique solution of the Cauchy problem

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)) \\ y(0) = x. \end{cases}$$

We aim to minimize the cost

$$J(x, \alpha) := \int_0^\infty \ell(y_x(t), \alpha(t)) e^{-\lambda t} dt.$$

For that purpose we define the value function as

$$v(x) := \inf_{\alpha \in \mathcal{A}} J(x, \alpha).$$

Prerequisites

Let $A \subset \mathbb{R}^M$ compact.

(A$_0$)

$$\begin{cases} A \text{ is a topological space,} \\ f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ is continuous,} \end{cases}$$

(A_1) f is bounded on $B(0, R) \times A$ for all $R > 0$,

(A_2) there is a modulus ω_f such that

$$|f(y, a) - f(x, a)| \leq \omega_f(|x - y|, R),$$

for all $x, y \in B(0, R)$ and $R > 0$.

(A$_3$)

$$(f(x, a) - f(y, a)) \cdot (x - y) \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R}^N, a \in A.$$

(A_4)

- ℓ is continuous,
- there are modulus ω_ℓ and a constant M such that

  $$|\ell(x, a) - \ell(y, a)| \leq \omega_\ell(|x - y|)$$

  and

  $$|\ell(x, a)| \leq M,$$

  for all $x, y \in \mathbb{R}^N$ and $a \in A$,
- $\lambda > 0$

Exercises

Exercise 1. Prove: Assume (A_0), (A_1), (A_3), and (A_4). Then $v \in BUC(\mathbb{R}^N)$. If moreover $\omega_\ell(r) = L_\ell r$ (i.e., ℓ is Lipschitz in y, uniformly in a), then v is Hölder continuous with the following exponent γ:

$$\gamma = \begin{cases} 1 & \text{if } \lambda > L \\ \text{any } \gamma < 1 & \text{if } \lambda = L \\ \frac{\gamma}{L} & \text{if } \lambda < L \end{cases}$$

Hint: Under the assumptions from above a basic property of y_x is

$$|y_x(t,\alpha) - y_z(t,\alpha)| \le e^{Lt}|x - z|$$

for all $\alpha \in \mathcal{A}$, and $t > 0$. L denotes the constant known from (A_3).

(6 Punkte)

Feedback maps

Definition 1. A control law or presynthesis on a set $\Omega \subseteq \mathbb{R}^N$ is a map $\mathbb{A} : \Omega \to \mathcal{A}$, that is, it associates with each point $x \in \Omega$ a control function $\mathbb{A}(x) =: a_x$. It is optimall on Ω if the cost associated with it, that is, $J_{\mathbb{A}}(x) := J(x, \alpha_x)$, satisfies

$$J_{\mathbb{A}}(x) = \min_{\alpha \in \mathcal{A}} J(x,\alpha) = v(x) \quad \text{for all } x \in \Omega.$$

The most important examples of control laws are generated by feedback maps $\Psi : \Omega \to A$, provided the feedback is admissible in the following sense.

Definition 2. A feedback map on a set $\Omega \subseteq \mathbb{R}^N$, $\Psi : \Omega \to A$, is admissible if for all $x \in \Omega$ there exists a unique solution $y_x(\cdot, \Psi)$ on $[0, +\infty[$ of

$$\begin{cases} \dot{(y)} = f(y, \Psi(y)) \\ y(0) = x \end{cases}$$

such that $t \to \Psi(y_x(t,\Psi))$ is measurable and $y_x(t,\Psi) \in \Omega$ for all $t \ge 0$.

It is natural to associate the following control law with an admissible feedback map

$$\alpha_x(\cdot) := \Psi(y_x(\cdot,\Psi)) \in \mathcal{A};$$

α_x in this case is called a closed-loop control.

Exercise 2. Denote by $\mathcal{F}$ the set of admissible feedback maps on $\mathbb{R}^N$, and set, for $\Psi \in \mathcal{F}$,

$$J_{\mathcal{F}}(x,\Psi) := \int_0^\infty e^{-\lambda t}\ell(y_x(t,\Psi), \Psi(y_x(t,\Psi)))dt$$

$$v_{\mathcal{F}}(x) = \inf_{\Psi \in \mathcal{F}} J_{\mathcal{F}}(x,\Psi).$$

(i) Prove that $v_{\mathcal{F}} = v$. [Hint: one inequality is trivial, the other is easily obtained by adding time as a state variable.]

(ii) Prove directly (without using (i)) that $v_{\mathcal{F}}$ is continuous and satisfies the Dynamic Programming Principle.

(6 Punkte)

Treating the convex HJ equation in multiple space dimensions

We consider Hamilton-Jacobi equations in multiple space dimensions

$$u_t + H(u_{x_1}, \dots, u_{x_d}) = 0, \quad \mathbb{R}^d \times [0,T] \tag{1}$$

Exercise 3. Generate the upwind scheme for (1) in case $d = 2$. Prove monotonicity and consistency for that scheme. (Adapt the univariate proofs).

(6 Punkte)