Exam: Geometry Processing & Discrete Shells

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Guiding questions and keywords

Introduction (Sec. 1)

- What's the Euler characteristic and how can it be used to derive mesh statistics?

Differential geometry of parametric surfaces (Sec. 2)

- Explain the parametric representation of a surface (*i.e.* parametrization, tangent spaces, etc)
- Define and explain the first/second FF resp. the shape operator! Comment on the differences of embedded and pull-backed quantities (*i.e.* representation in parameter domain)! What are crucial properties and relations between these objects?
- What notions of curvature do you know? Can you give examples? (no normal curvature!)
- State the fundamental theorem of surfaces and explain the implications on the (discrete) shell space.
- Why do we need a pulled-back resp. a relative shape operator?
- How are differential operators defined on a surface? (Rather idea/motivation than formula!)
- How can the Laplace-Beltrami operator of the embedding be represented?
- What are intrinsic and extrinsic quantities, respectively? Can you give examples?
- State the Theorema Egregrium/Gauss-Bonnet-Theorem! Explain their meaning and their impact on the tpoic of the course.
- Coordinate-based calculations, *e.g.* $g_{ij}(\xi) = x_{,i}(\xi) \cdot x_{,j}(\xi) \Rightarrow \partial_{\xi_k} g_{ij}(\xi) = x_{,ik}(\xi) \cdot x_{,j}(\xi) + x_{,i}(\xi) \cdot x_{,jk}(\xi)$

Discrete differential geometry (Sec. 3)

- What are the guiding principles of DDG? Can you give examples how they have been used in the lecture course?
- What is a discrete surface?
- How does the parametrization of a mesh work?
- Derive the discrete fFF!
- Discrete Laplace-Beltrami:
  - Explain the concept of weak derivatives on a surface and define Sobolev spaces on a surface.
  - Give the definition of the weak Laplace-Beltrami.
  - Explain the basic concepts of linear FEM on a mesh (*e.g.* basis functions, gradients, discrete (nodal) functions).

- Derive the discrete weak Laplace-Beltrami and the cotan formula via geometric arguments or via linear FEM! How do the two approaches relate?
- How do we obtain a pointwise discrete Laplace-Beltrami? What are the differences w.r.t. the weak formulation?

- Convergence of discrete mean curvature
  - What can we learn from Schwartz lantern example?
  - What's the definition of the shortest distance map and why is it needed?
  - Define the metric distortion tensor and state the splitting theorem! What is the intuition and what are implications? Explain the basic idea of the proof!
  - What is totally normal convergence? Are there equivalent conditions?
  - Define the pulled-back discrete Laplace-Beltrami and the discrete mean curvature functional! What's the relation to the cotan-formula?
  - What do we know about convergence of mean curvature? How does the proof work?
  - Explain the difference between weak/integrated/functional/distributinal sense and pointwise/function sense in terms of properties/convergence/applications.
  - Can we expect pointwise or L^2 convergence of the mean curvature function?

- Notions of discrete curvature measures
  - Give examples for different discrete normal fields on a mesh.
  - What kind of vertex-based measures do you know? What's the motivation for the angle defect definition?
  - What's the dihedral angle? How is is motivated by Steiner's tube formula?
  - What are the main results of Cohen-Steiner and Morvan?
  - What's the conncetion of edge-based measures to non-conforming FEM (*e.g.* mean curvature)?
  - Explain the main idea and motivation for the triangle-averaged shape operator? How is the differential of the discrete normal evaluated on a mesh?
  - Derive the formula for a triangle-averaged mean curvature based on the dihedral angle. Is there a relation to non-conforming FEM?

Deformations of discrete shells (Sec. 4)

- Why do we need a physical model at all?
- What is a thin shell?
- Write down a typical hyperelastic energy and explain usual assumptions!
- What are the implications of the Rivlin-Erikson Thm.?
- Explain the (desired) properties of a typical elastic energy density.
- How do we obtain a linear elastic model?
- Explain the role of rigid body motion invariance! Why is it important for our purposes? Is it given in the nonlinear/linear regime?
- How do we get a 2D model out of a 3D model?
- What are the hypotheses of Reissner-Mindlin? How do they restrict the space of admissible deformations/displacements?
- What's the Kirchhoff-Love hypothesis? Explain the relation to the Reissner-Mindlin model!
- Derive the linear plate model for the Reissner-Mindlin model!

- How do we obtain the Kirchhoff-Love plate model? How can we avoid a H^2-conforming approach in the discretization?
- Explain the concept of Γ-convergence! What are basic properties? Can you give examples? Why is it a usefull concept in the context of calculus of variations?
- Give examples for different boundary conditions for a thin plate deformation and comment on the scaling of the elastic energy!
- Why do we need a resclaing of the elastic energy?
- Explain the qualitative properties of the membrane Γ-model. How do we represent the geometric distortion tensor?
- Explain the qualitative properties of the bending Γ-model for plates. What terms do we see in the limit?
- How can one construct a recovery sequence? Why do we get infinite energy for non-isometric deformations?
- Explain the qualitative properties of the bending Γ-model for shells. What terms do we see in the limit?
- State the full elastic shell model that we have derived from the Γ-limits! Comment on the resulting dissimilarity measure!
- What is dense correspondence and why is it needed?
- Derive the discrete membrane and bending model!
- How can we obtain a simplified discrete bending model (starting from the full model)? What are the assumptions/approximations used here?

The shape space of discrete shells (Sec. 5)

- Why do we consider the shape space as a Riemannian manifold? How can one derive a Riemannian structure in general? How are the approaches related?
- Explain different definitions of geodesic paths! How are they related?
- Define the length and the path energy of a curve and comment on properties and differences!
- How can we represent the Christoffel operator (coordinate-free)?
- What is the covariant derivative? Can you derive it?
- Explain the concept of parallel transport and its relation to geodesic paths?
- What is the exponential map? Is it invertible?
- How do we discretize geodesic path in time?
- Explain the motivation for the discrete length and the discrete path energy!
- What is a local approximation of the squared Riemannian distance and what is it needed for? How does it relate to the metric?
- What's the relation of minimizers of the discrete length and the discrete path energy, respectively, to continuous geodesics?
- How do we compute discrete geodesics in the discrete shell space? Write down the necessary conditions!
- Explain the motivation for the discrete logarithm and the discrete exponential! Why is the discrete exponential map consistent with the notion of discrete geodesics?
- Explain the idea of Schild's ladder! How can the concept of a parallelogram be transferred to a Riemannian setup and a discrete Riemannian setup?

A selection of suitable topics to start with (among others!)

- derivation of cotan formula as discretization of weak Laplace-Beltrami
- convergence results for discrete mean curvature
- motivation of edge-based curvature measures by Steiner's tube formula
- derivation of triangle-averaged discrete shape operator
- derivation of Reissner-Mindlin 2D model from linear elasticity
- derivation of nonlinear 2D shell model via Γ-convergence results
- derivation of simplified discrete bending model
- derivation/motivation of time-discrete geodesics on a generic manifold
- derivation of further geoemtric objects, *e.g.* discrete parallel transport