Geometry Processing & Discrete Shells

(V5E2)

3D acquisition technology (e.g. computer tomography, MRI, 3D laser scanning)

* neuroscience
* medicine
* mech. engineering
* material sciences
* architecture
* geo-exploration
* entertainment

analysis & processing

digital manufacturing; (e.g. 3D printing) novel materials & robotic production by digital design plan

raw geometric material ← GP → production based on geometric models

Here: focus on data describing the surface / boundary of a compact 3D solid, which is digitally represented as a polygon mesh (e.g. triangle meshes)

Typical problems in GP:

* smoothing & filtering (removal of topolog. / geom. errors, noise
* surface reconstruction (to apply mesh based algo.)
* remeshing (to increase mesh quality)
* model repair (remove artifacts like holes)
* simplification & approximation (reduction of complexity)
* deformation, modeling & interactive design

Main focus of this course:

* mathematical foundation of deformations & modeling tasks of complex geometric objects represented as triangle meshes
* (assume that nice triangulation is already given)
* simpl. & approx to enable efficient algo. (multiresolution)

Key words:

(a) local geometry: geom. quantities on a single mesh, in particular notion of discrete curvature & convergence analysis

(b) physics: nonlinear deformations based on physically sound thin shell model (discrete shells)

(c) global geometry & dynamics: each mesh is single point in high-dim. shape space; (Riemannian manifold)

Outline:

1. Surface representations
   * implicit vs. explicit
   * examples

2. Diff. geometry of parametric surfaces
   * first & second fund. forms
   * (relative) shape operator
   * differential operators, e.g. Laplace-Beltrami

3. Discrete diff. geometry
   * discrete Laplace-Beltrami
   * discrete curvature measures
   * convergence
   * applications

4. Deformations of discrete shells
   + physics of thin shells
   + discrete deformation energies

5. Implementation of deformation techniques
   + variational techniques & multiresolution
   + differential representations

6. The shape space of discrete shells
   + variational time-discretization
   + navigation in the space of DS (interpol., extrapol., ...)

Oral exam: 27.7.17
(registration in BASIS → 29.6.17)
(sign off → 20.7.17)

Points to discuss:

+ Bachelor vs. master
+ Background in:
  (i) diff. geom. (of surfaces)
  (ii) numerics, FEM
  (iii) Functional analysis, Sobolev spaces (weak deriv.)
  (iv) programming
+ primal focus: mathematical modeling & applications vs. convergence analysis & theory
+ timing of lecture (break?)

+ language
+ references
+ notes / script
+ homepage
+ email.

1. Surface representations

"an orientable, compact, regular 2D-manifold embedded in $\mathbb{R}^3$, which might be closed or not"

* (part of) boundary of a non-degenerate 3D solid
* [non-deg.: no infinitesimal thin parts/features, surface separates properly interior/exterior]

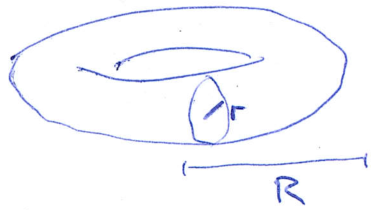
1.1 Implicit vs. explicit representations

Explicit/parametric: there is a parametrization f over a parameter domain $\Omega \subset \mathbb{R}^2$, i.e. $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$, and $S = f(\Omega)$.

Implicit: (zero-) levelset of a scalar-valued function $F: \mathbb{R}^3 \to \mathbb{R}$, $S = \{x \in \mathbb{R}^3 : F(x) = 0\}$.

Example: Torus, $0 < r < R$

$$f(\theta, \phi) = \begin{pmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta \end{pmatrix}, \quad (\theta,\phi) \in [0, 2\pi)^2$$



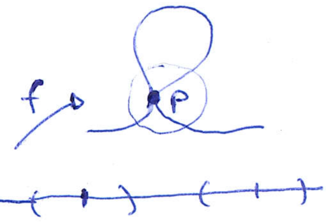
$$F(x,y,z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2+y^2)$$

Note: often necessary to split domain in several patches; $\to$ need to guarantee consistent/smooth transition!

Regularity:

Def: A parametric surface S is locally manifold at $p \in S$, if $\exists\, \varepsilon > 0$ such that $f^{-1}(B_\varepsilon(p))$ is an open ball in Ω.

One also says that the neighborhood of p is topologically equivalent (homeomorphic) to a disk.



Def: A surface is of class C^k if $f \in C^k(\Omega)$ resp. $F \in C^k(\mathbb{R}^3)$

Fairness: A (discrete) surface is considered as fair, if the curvature (or its variation) is globally minimized.
[no rigorous definition, rather aesthetic concept]

Parametric vs. Implicit:

[The two representations have] almost complementary strengths & weaknesses.

Investigate performance of typical operations:

* evaluation (of points, normals, ..)
* (spatial) queries, (inside/outside)
* modification of geometry or topology.

Parametric surfaces:

+ reduction of many 3D problems to $\Omega \subset \mathbb{R}^2$
+ capture finest geometric details
+ easy to sample
+ efficient shape editing, $\Phi \circ f$, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ $*$
- generation of parametric surface expensive
- change of topology expensive
- inside/outside resp. signed distance expensive
- detection of self-collision hard

Implicit surfaces:

+ no holes if $F \in C^0$
+ no geometric self-intersections
+ signed-distance queries easy (if $S = [F = 0]$)
+ construction of complex objects easy
- geometric detail resolution depends on voxel-size
- sampling difficult.
- surface editing expensive

In this course: parametric surfaces! cf. $*$

1.2 Approximation power:

* natural choice for parametrizing functions are polynomials
* Taylor: On interval of size h any smooth function can be approx. by a polynomial of degree p with $O(h^{p+1})$
* p-refinement vs. h-refinement
* in GP: usually h-refinement, i.e. $p = 1$ & $h \to 0$
(processing large number of simple objects easier than small number of large objects

1.3 Examples of parametric representations

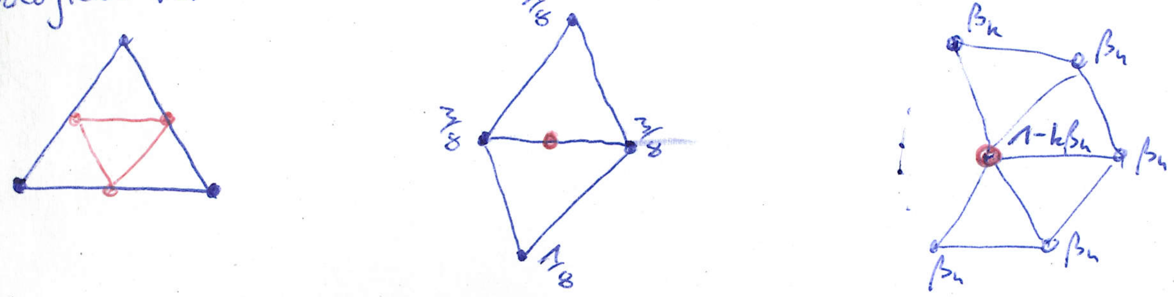
Ex 1 (Subdivision surfaces)

- Starting with coarse mesh S_0 with vertices $c_1, \dots, c_n \in \mathbb{R}^3$ (control mesh)

- generate sequence of meshes $S_1, S_2, \dots$ with growing complexity by subdivision rules

Loop subdivision: For $n = 1, 2, \dots$

Topological refinement: Geometrical ~~refinement~~ update:



$n=6$: $\beta_n = \frac{1}{16}$

Loop: $\beta_n = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\frac{2\pi}{n}\right)^2\right)$

Thm: $S_n \to S_\infty$, $S_\infty \in C^2$ except for a finite set of points, where S is of class C^1.

S_∞ can be parametrized over S_0 by (cubic) spline functions $\phi_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$:

$$f(u,v) = \sum_{i=1}^{n} c_i \, \phi_i(u,v)$$

$\leadsto$ intuitive control of surface via $(c_i)_i$. (CAD)

Adaptivity:

$$\phi_i(u,v) = \phi_i^0(u,v) = \sum_j d_{ij}^k \, \phi_j^k(u,v)$$

$\leadsto$ inherent hierarchical structure!

Ex2: (Unstructured triangle meshes)

* allow for arbitrary connectivities
* geometric details can be represented easily
* highly flexible, still efficient processing

Triangle mesh M_h = geom. $\oplus$ topology (connectivity)
repres. as graph structure

$\mathcal{V} = \{v_1, \dots, v_n\}$

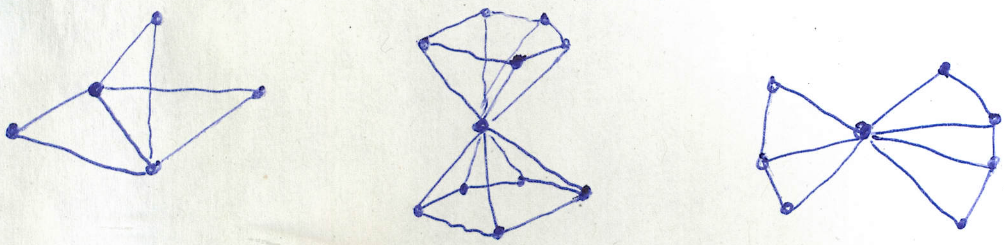
$\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V}$

$(\Rightarrow \mathcal{E} \subset \mathcal{V} \times \mathcal{V})$

$\oplus$ geometric embedding: $E : \mathcal{V} \to \mathbb{R}^3$, $E(v_i) = X_i$.

Assumption: M_h is discrete 2-manifold, i.e. no

* non-manifold edges
* non-manifold vertices
* self-intersections (locally manifold)



Euler formula: M_h closed & connected mesh:

$$|\mathcal{V}| + |\mathcal{F}| - |\mathcal{E}| = 2(1-g) \approx 0$$

Since $3|\mathcal{F}| = 2|\mathcal{E}|$ we can deduce:

* $|\mathcal{F}| \approx 2|\mathcal{V}|$
* $|\mathcal{E}| \approx 3|\mathcal{V}|$
* average valence is 6

In this course:
unstructured triangle meshes

Repetition:

We consider unstructured triangle meshes $\mathcal{M}_h$,
$\mathcal{M}_h = \{\mathcal{V}, \mathcal{F}, \mathcal{E}\}$ plus embedding $E: \mathcal{V} \to \mathbb{R}^3$.

Assumption: $\mathcal{M}_h$ is discrete 2-manifold.

Euler formula (for a closed mesh)

$$|\mathcal{V}| + |\mathcal{F}| - |\mathcal{E}| = 2(1-g)$$

Furthermore: $|\mathcal{E}| = (3 \cdot |\mathcal{F}|) \cdot \frac{1}{2} \Rightarrow 2|\mathcal{E}| = 3|\mathcal{F}|$

2. Differential geometry of parametric surfaces

References: • M. do Carmo, "Differential geom. of curves and surfaces", 1976

• C. Bär, "Elementare Differentialgeometrie", 2000

Definition 2.1 (Regular surface)

The set $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$ there is $\varepsilon > 0$, an open set $\Omega \subset \mathbb{R}^2$ and a smooth mapping $x: \Omega \to \mathbb{R}^3$, such that

(i) $x(\Omega) = S \cap B_\varepsilon(p)$ and $x: \Omega \to S \cap B_\varepsilon(p)$ is homeomorphic

(ii) $\operatorname{rank} D_\xi x = 2 \quad \forall \xi \in \Omega \quad (D_\xi x \in \mathbb{R}^{3,2})$

Remark: We shall assume we have a global parametrization, i.e. $x: \Omega \to S$.

Let $\xi = (\xi_1, \xi_2) \in \Omega$, $p = x(\xi) \in S$.

$$T_pS = \{ \dot{\gamma}(0) \mid \gamma : (-1,1) \to S,\ \gamma(0) = p \}$$

$$T_\xi \Omega = \{ \dot{\alpha}(0) \mid \alpha : (-1,1) \to \Omega,\ \alpha(0) = \xi \} \equiv \mathbb{R}^2$$

If $\alpha : (-1,1) \to \Omega$, $\alpha(0) = \xi$, we set $\gamma_\alpha := x \circ \alpha$.

Then $\dot{\gamma}_\alpha(0) = Dx \cdot \dot{\alpha}(0)$ and

$$T_pS = Dx\, T_\xi \Omega = \operatorname{span}\{ V_1, V_2 \}$$

where

$$V_i := \partial_i x(\xi) = \partial_{\xi_i} x(\xi) = \frac{\partial x(\xi)}{\partial \xi_i} = Dx\, e_i \ , \quad i = 1,2,$$

is the <u>canonical basis</u> of T_pS; $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$.

2.1 Fundamental forms

<u>Def. 2.2</u> (First fund. form)

The first fund. form in $p \in S$ is given by

$$g_p : T_pS \times T_pS \to \mathbb{R}, \quad g_p(U,V) := \langle U, V \rangle_{\mathbb{R}^3}.$$

Represent g_p in basis (V_1, V_2) as a matrix $g \in \mathbb{R}^{2,2}$:

$$g_{ij} := g_p(V_i, V_j) \quad \Rightarrow \quad g = Dx^T Dx$$

The pull-back of g_p to Ω is defined by

$$g_\xi(u,v) = g_p(Dx\,u, Dx\,v) = u^T g v, \quad u, v \in T_\xi\Omega = \mathbb{R}^2.$$

Geometrically: 1FF measures length & angles.

Let $\gamma_\alpha = x \circ \alpha$.

$$\mathcal{L}[\gamma_\alpha] = \int_{-1}^{1} |\dot{\gamma}_\alpha(t)|\, dt = \int_{-1}^{1} \sqrt{\langle Dx\,\dot{\alpha}(t), Dx\,\dot{\alpha}(t)\rangle}\, dt = \int_{-1}^{1} \sqrt{g_{\alpha(t)}(\dot{\alpha}(t), \dot{\alpha}(t))}\, dt$$

Notation: $g \hat{=} g_\xi \hat{=} g_p$; $g \hat{=}$ matrix ; $g \hat{=}$ bilinear form

Remark: $g \in \mathbb{R}^{2\times 2}$ is invertible since S is regular.

Let $A \subset \Omega$, $\ell : S \to \mathbb{R}$.

$$\int_{x(A)} \ell \, da = \int_A (\ell \circ x)(\xi) \sqrt{\det g_\xi}\, d\xi$$

For $\ell \equiv 1$: $\quad vol(x(A)) = \int_A \sqrt{\det g_\xi}\, d\xi .$

Differentiation: For $\ell : S \to \mathbb{R}$ we define the differential $d_p \ell$ as a linear form on $T_p S$, i.e. the directional derivative,

$$d_p \ell (V) = \frac{d}{dt} \ell(\gamma(t)) \Big|_{t=0}$$

for $\gamma : (-1,1) \to S$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$.

For a vector-valued deformation $\Phi : S \to \mathbb{R}^3$, $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, definition holds for each Φ_i. In particular:

$$d_p \Phi : T_p S \to T_{\Phi(p)} \Phi(S).$$

Definition 2.3 (Normal field)

The unit normal field of S is a mapping $n : S \to S^2$ with $n(p) \perp T_p S$ for all $p \in S$, i.e.

$$n(p) = (n \circ x)(\xi) = \frac{V_1 \times V_2}{|V_1 \times V_2|}(\xi)$$

S is orientable if there is a continuous normal field.

Definition 2.4 (Shape operator)

The shape operator $S_p : T_pS \to T_pS$ is a linear mapping with
$$S_p(U) := d_p n(U), \quad U \in T_pS.$$

Remark: $T_{n(p)}S^2 = n(p)^\perp = T_pS. \Rightarrow S_p$ endomorphism.

Definition 2.5 (Second fund. form)

The second fund. form $h_p : T_pS \times T_pS \to \mathbb{R}$ is given by
$$h_p(U,V) = g_p(S_pU, V), \quad U,V \in T_pS.$$

The matrix representation wrt. (V_1, V_2) is given by
$$h_{ij} := h_p(V_i, V_j) = g_p(S_pV_i, V_j) = \langle \partial_i(n \circ x), \partial_j x \rangle_{\mathbb{R}^3}$$

Hence $h = h_\xi \in \mathbb{R}^{2,2}$. Since $n(p) \perp T_pS$:
$$0 = \partial_{\xi_i}\left(g_p(n \circ x, \partial_j x)\right) = h_{ij} + g_p((n \circ x), \partial_{ij}^2 x)$$

This implies h is symmetric and
$$h_{ij} = -\langle n \circ x, \partial_{ij}^2 x \rangle_{\mathbb{R}^3}$$

Remark: We often write $h_\xi = Dn^T Dx$, $Dn = \left[\frac{\partial n(p)}{\partial \xi_1}, \frac{\partial n(p)}{\partial \xi_2}\right] \in \mathbb{R}^{3,2}$.

Represent S_p in basis (V_1, V_2):
$$S_p V_i = \sum_{k=1}^{2} s_{ki} V_k$$

$$\Rightarrow h_{ij} = g_p(S_pV_i, V_j) = \sum_{k=1}^{2} s_{ki} \underbrace{g_p(V_k, V_j)}_{g_{kj}} = (s^T g)_{ij}$$

Since g, h are symmetric:
$$s_\xi := g_\xi^{-1} h_\xi$$

Notation: $S_p : T_pS \to T_pS$, $S_p \in \mathbb{R}^{3,3}$, $s = s_\xi \in \mathbb{R}^{2,2}$.

S_p is symmetric wrt. g_p:

$$g_p(S_pU, V) = h_p(U,V) = h_p(V,U) = g_p(S_pV, U)$$
$$= g_p(U, S_pV).$$

$\Rightarrow$ S_p (and s_ξ) diagonalize in an orthonormal basis.

Definition 2.6 (Curvatures)

The eigenvalues $\kappa_1, \kappa_2 \in \mathbb{R}$ of s_ξ are denoted as principal curvatures of S in $p = x(\xi)$.

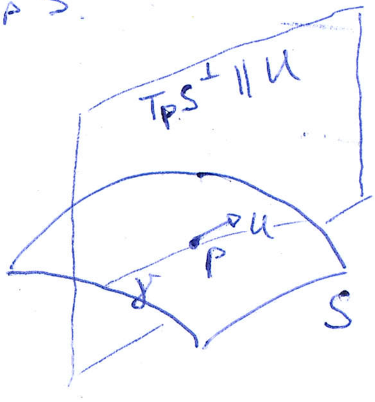
$H_p := \operatorname{tr} s_\xi = \kappa_1 + \kappa_2$ is mean curvature

$K_p := \det s_\xi = \kappa_1 \cdot \kappa_2$ is Gaussian curvature.

Remark: $\det S_p = 0$ $\Rightarrow$ eigenvalues of S_p are $0, \kappa_1, \kappa_2$.

Remark: The normal curvature κ_p in $p \in S$ is given by

$$\kappa_p(U) := \frac{h_p(U,U)}{g_p(U,U)}, \quad U \in T_pS.$$



$\kappa_p(U)$ describes curvature of $\gamma : (-1,1) \to S$ with $\gamma(0) = p$ and $\dot\gamma(0) = U$.

If $\kappa_1 \le \kappa_2$:

$$\kappa_1 = \min_{U \in T_pS} \kappa_p(U), \quad \kappa_2 = \max_{U \in T_pS} \kappa_p(U).$$

2.2. The relative shape operator

Later: measure differences between two (discrete) surfaces $S, \tilde{S}$ up to rigid body motions.

Theorem 2.7 (Fund. theorem of surfaces)

Congruent parametric surfaces in $\mathbb{R}^3$ have the same first & second fund. forms. Conversely, two parametric surfaces in $\mathbb{R}^3$ with the same first & second fund. forms are congruent.

$$S \overset{\wedge}{\equiv} \tilde{S} \quad \Longleftrightarrow \quad \left\{ \begin{matrix} g \equiv \tilde{g} \\ h \equiv \tilde{h} \end{matrix} \right\}$$

congruent; up to rbm.

$\Rightarrow$ measure differences in $g \leftrightarrow \tilde{g}$ and $h \leftrightarrow \tilde{h}$.

For isometric deformations, i.e. $\tilde{S} = \Phi(S)$, Φ isometric, we have $g = \tilde{g}$ and study differences in shape operators.

However: $S_p : T_pS \to T_pS$ but $\tilde{S}_{\tilde{p}} : T_{\tilde{p}}\tilde{S} \to T_{\tilde{p}}\tilde{S}$.

Definition 2.8 (Pulled-back shape op.)

$$S_p^*[\Phi] : T_pS \to T_pS, \quad g_p(S_p^*[\Phi]U, V) := h_{\Phi(p)}(D\Phi U, D\Phi V)$$
$$\forall U, V \in T_pS.$$

Definition 2.9 (Relative shape op.)

$$S_p^{rel}[\Phi] : T_pS \to T_pS, \quad S_p^{rel}[\Phi] := S_p - S_p^*[\Phi].$$

Later: physical/analytical argument why to study $S_p^{rel}[\Phi]$.

2.3 Differential operators

For $f: \Omega \subset \mathbb{R}^d \to \mathbb{R}$:

$$\langle \nabla f(x), W \rangle_{\mathbb{R}^d} = \frac{d}{dt} f(x+tW)\Big|_{t=0} \quad \forall W \in T_p\Omega = \mathbb{R}^d.$$

Now: $f: S \subset \mathbb{R}^3 \to \mathbb{R}$. For $p \in S$ we have $\nabla_S f(p) \in T_pS$.

Write $\nabla_S f(p) = \sum_i v_i V_i = Dx\, v$ for $v = v_f \in \mathbb{R}^2$.

Definition: $$g_\xi(v,w) := \frac{d}{dt} \tilde{f}(\xi + tw)\Big|_{t=0} = \langle \nabla \tilde{f}(\xi), w \rangle_{\mathbb{R}^2}$$

$$\forall w \in T_\xi \Omega.$$

with $\tilde{f} = f \circ x : \Omega \to \mathbb{R}$.

$\Rightarrow v = v_f = g_\xi^{-1} \nabla \tilde{f} \quad \Rightarrow \boxed{\nabla_S f = Dx\, g^{-1} \nabla (f \circ x).}$

Next: vector field w on S, $w(p) \in T_pS$ for each $p \in S$.

Set $\tilde{w} = w \circ x$, let $e \in C^\infty(S)$ s.t. $\tilde{e} = e \circ x \in C^\infty(\Omega)$,

let $v_{\tilde{e}} = g^{-1} \nabla \tilde{e}$.

$$\left\{ \begin{array}{l} \text{For vector field } w: \Omega \subset \mathbb{R}^d \to \mathbb{R}^d \text{ we have} \\ \displaystyle\int_\Omega \operatorname{div} w \cdot e \, dx = - \int_\Omega \langle w, \nabla e \rangle_{\mathbb{R}^d} \, dx \quad \forall e \in C_0^\infty(\Omega). \end{array} \right\}$$

Then

$$\int_\Omega \operatorname{div}_S \tilde{w} \cdot \tilde{e} \sqrt{\det g_\xi}\, d\xi := - \int_\Omega \underbrace{g_\xi(\tilde{w}, v_{\tilde{e}})}_{= \langle \tilde{w}, \nabla \tilde{e} \rangle_{\mathbb{R}^2}} \sqrt{\det g_\xi}\, d\xi$$

$$\underset{\text{[integration by parts]}}{=} \int_\Omega \operatorname{div}\left(\sqrt{\det g_\xi}\, \tilde{w}\right) \cdot \tilde{e}\, d\xi$$

Hence:

$$(\mathrm{div}_S w) \circ x = \frac{1}{\sqrt{\det g}} \mathrm{div}\left(\sqrt{\det g}\,(w \circ x)\right)$$

<u>Definition 2.10</u> (Laplace-Beltrami operator)
For some function $f: S \to \mathbb{R}$, the Laplace-Beltrami-op. is given by $\Delta_S f = \mathrm{div}_S \nabla_S f$, i.e.

$$(\Delta_S f) \circ x = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{2} \partial_{\xi_i}\left(\sqrt{\det g} \sum_{j=1}^{2} g^{ij} \partial_{\xi_j}(f \circ x)\right),$$

with $\sum_j g_{ij} g^{jh} = \delta_{ih}$, i.e. $g^{-1} = (g^{ij})_{ij}$.

<u>Theorem 2.11</u>. Let $id: S \to \mathbb{R}^3$, i.e. $id(p) = p \;\forall p \in S$, then we have $\quad (\Delta_S\, id)(p) = -H_p\, n(p)$.

<u>2.4 Intrinsic vs. extrinsic surface quantities</u>

<u>Definition</u> (Local isometry)
$\Phi: S \to \tilde{S}$ is local isometry, if for each $p \in S$:

$$g_p(V, W) = \tilde{g}_{\tilde{p}}(d_p\Phi(V), d_p\Phi(W)) \quad \forall V, W \in T_pS.$$

where $\tilde{p} = \Phi(p) \in \tilde{S}$.

A quantity, i.e. $f_S: S \to \mathbb{R}$, is said to be <u>intrinsic</u>, if $\quad f_S = f_{\tilde{S}} \circ \Phi \quad \forall$ local isometries $\Phi: S \to \tilde{S}$.

Intrinsic quantities : g, Δ_S, da, K_p

Extrinsic quantities : $\kappa_1, \kappa_2, H_p, h, S_p$

Theorem 2.13 (Theorema egregium, Gauss)

The Gaussian curvature is an intrinsic surface property.

Remark : A sphere cannot be unfolded onto a flat plane without distorting distances (map of Earth!)

Later : Want to quantify distortions induced by isometric deformations $\Rightarrow$ use extrinsic quantities

One more interesting result on Gaussian curvature :

Theorem 2.14 (Thm. of Gauss-Bonnet)

For a compact, orientable, closed surface $S \subset \mathbb{R}^3$:

$$\int_S K \, da = 2\pi \cdot \chi_S ,$$

where $\chi_S := 2(1-g)$ is Euler characteristic, g is genus.

If we approximate $S \approx \mathcal{M}_h = (\mathcal{V}, \mathcal{F}, \mathcal{E})$:

$$\int_S K \, da = 2\pi \cdot \left(|\mathcal{V}| + |\mathcal{F}| - |\mathcal{E}| \right) .$$

Add on:

$$S_p^*[\Phi]: T_pS \to T_pS, \quad g_p(S_p^*[\Phi]u, v) = h_{\Phi(p)}(D\Phi u, D\Phi v)$$

$$S_p^{rel}[\Phi]: T_pS \to T_pS, \quad S_p^{rel}[\Phi] = S_p - S_p^*[\Phi]$$

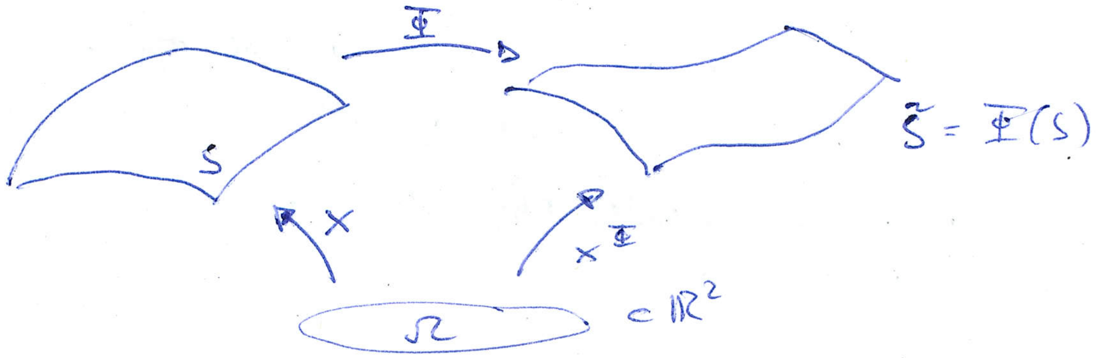
The matrix representations $s_\xi^*[\Phi], s_\xi^{rel}[\Phi] \in \mathbb{R}^{2,2}$ are given by

$$s_\xi^*[\Phi] = g_\xi^{-1} h_\xi^\Phi,$$

$$s_\xi^{rel}[\Phi] = s_\xi - s_\xi^*[\Phi] = g_\xi^{-1}(h_\xi - h_\xi^\Phi).$$

Note: If $x: \Omega \to S$ is a parametrization of S, then $x^\Phi = \Phi \circ x: \Omega \to \Phi(S)$ is a parametrization of $\Phi(S)$ over the same reference domain.

Here h^Φ denotes the SFF w.r.t. $\Phi(S)$ resp. x^Φ.



$$x^\Phi = \Phi \circ x^\Phi$$

Repetition:

FFF: $g_p(u, v) = \langle u, v\rangle_{\mathbb{R}^3}$

$g = Dx^T Dx \in \mathbb{R}^{2,2}$ invertible.

$S_p: T_pS \to T_pS, \quad S_p u := d_p n(u), \quad u \in T_pS.$

SFF: $h_p(u, v) = g_p(S_p u, v)$

$h = Dn^T Dx \in \mathbb{R}^{2,2}$ symmetric

$s = g^{-1}h \in \mathbb{R}^{2,2}$ is matrix repres. of shape op. in Ω.

Differential ops:

$$\nabla_S f = Dx\, g^{-1} \nabla (f \circ x)$$

$$\mathrm{div}_S\, w = \det g^{-\frac{1}{2}} \,\mathrm{div}\left(\det g^{-\frac{1}{2}} (w \circ x)\right)$$

$$\Delta_S f = \mathrm{div}_S \nabla_S f.$$

Thm 2.11: $(\Delta_S\, id) = -Hn.$

3 Discrete differential geometry

Problem: differential / geometric quantities defined on $S \subset \mathbb{R}^3$ require some regularity, e.g. curvatures need existence of 2^{nd} derivatives (of x, cf. def. of h)

However, a triangle mesh is pw. affine hence of class C^0.

But: Since $M_h \approx S$ one aims to compute approx. of these quantities of S based on M_h, i.e. the mesh data.

⇝ derive discrete equivalents of geometric notions of classical diff. geometry which are consistent.

"Theories, which are

(i) discrete from the start and

(ii) satisfy key geometric properties built into their description

can often [more readily]* yield robust numerical simulations which are true to the underlying continuous system."

[Siggraph Asia Course Notes 08]

* in comparison to straight-forward discretizations of cont. equations.

Guiding principles of DDG:

* weak / integrated notions, e.g. $\Delta_h u \in H^{-1}$, convergence in weak / integrated sense

* spatial averaging to get pointwise evaluations (but: consistency & convergence lost!)

* attach quantities to appropriate locations, i.e.
  * functions $\hat{=}$ 0-forms at vertices
  * vector fields $\hat{=}$ 1-forms at edges, defined as line integrals

* Discretize theory, not equations! → aims at consistent discrete theory!
  Ex: Gauss-Bonnet.

3.1 Basic notions

triangular mesh M_h $\hat{=}$ geometry $\oplus$ topology / connectivity

$$E : \mathcal{V} \to \mathbb{R}^3 \qquad\qquad \begin{array}{l} \mathcal{F} \subset \mathcal{V} \times \mathcal{V} \times \mathcal{V} \\ \mathcal{E} \subset \mathcal{V} \times \mathcal{V} \end{array}$$

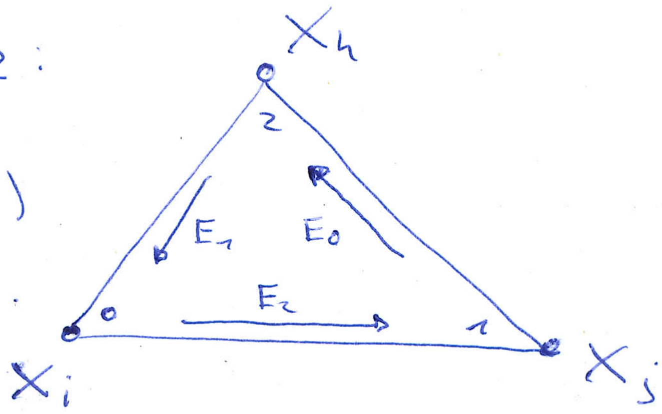
→ all further properties, e.g. neighboring relationships, boundary, etc. can be derived from $\mathcal{F}$ resp. $\mathcal{E}$.

Geometry: $n := |\mathcal{V}|$, $\mathcal{V} = \{v_1, \ldots, v_n\}$.
$m := |\mathcal{F}|$, $\mathcal{F} = \{f_1, \ldots, f_m\}$

$f_i = (v_{i_0}, v_{i_1}, v_{i_2})$, then embedded triangle

$$T_i = T(f_i) = (E(v_{i_0}), E(v_{i_1}), E(v_{i_2})) = (X_{i_0}, X_{i_1}, X_{i_2}) \in \mathbb{R}^3$$

Global vs. local indices:



$T = T(f) = (X_i, X_j, X_k)$

$= (X_0(f), X_1(f), X_2(f))$.

$E_i = X_{i-1} - X_{i+1}$

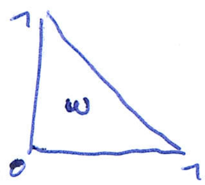
Topology: $\mathcal{M}_h$ is discrete 2-manifold,
e.g. no hanging nodes, no degenerated faces, no parallel edges, no self-intersections, ...

Orientation: defined by ordering of local indices (cw / ccw), assumed to be consistent!

Def. 3.1 (Discrete surface)
A discrete surface is a triangular mesh $\mathcal{M}_h = (\mathcal{V}, \mathcal{F}, \mathcal{E})$ with an injective embedding $E : \mathcal{V} \to \mathbb{R}^3$ such that $\mathcal{M}_h$ is a discrete 2-manifold which is orientable (i.e. has consistent local index ordering).

Remark: $\mathcal{M}_h = (E(\mathcal{V}), \mathcal{F})$, $E(\mathcal{V}) \in \mathbb{R}^{3n}$.

Parametrization: The reference domain Ω_h of $\mathcal{M}_h$ is a collection of m unit triangles $\omega := \left(\binom{0}{0}, \binom{1}{0}, \binom{0}{1} \right) \in \mathbb{R}^2$ and each face $f \in \mathcal{F}$ is parametrized over ω, via an affine mapping $X_f : \omega \to T(f)$,



$$X_f(\xi) := X_f(\xi_1, \xi_2) = \xi_1 X_1(f) + \xi_2 X_2(f) + (1 - \xi_1 - \xi_2) X_0(f)$$

for $\xi \in \omega$, i.e. $0 \le \xi_1, \xi_2 \le 1$, $\xi_1 + \xi_2 \le 1$.

Then we have $\Omega_h = \omega \times \mathcal{F}$, with a global parametrization

$$X : \Omega_h \to M_h,$$
$$(\xi, f) \mapsto X_f(\xi).$$

Discrete first fundamental form

Recall: $g_\xi = D_\xi X^T D_\xi X \in \mathbb{R}^{2,2}$

X affine $\Rightarrow$ $DX \in \mathbb{R}^{3,2}$ constant on faces.

$$DX|_f = \left[\frac{\partial X_f}{\partial \xi_1} \middle| \frac{\partial X_f}{\partial \xi_2} \right] = \left[X_1(f) - X_0(f) \,\middle|\, X_2(f) - X_0(f) \right]$$
$$= \left[E_2(f) \,\middle|\, -E_1(f) \right]$$

Hence:

$$G_f := (DX|_f)^T (DX|_f)$$

Note that $\det G_f = 0$ iff. $T(f)$ has parallel edges, which is forbidden since M_h is 2-manifold.

Finally we get

$$a_f := |T| = \int_T da = \int_\omega \sqrt{\det G_f}\, d\xi = \frac{1}{2} \sqrt{\det G_f}$$

3.2 Discrete Laplace - Beltrami

Def 3.2 (Weak derivatives)

Let $S \subset \mathbb{R}^3$ regular surface. Then $u: S \to \mathbb{R}$ has a weak surface gradient $\nabla_S u$ if

$$\int_S g(\nabla_S u, V)\, da = -\int_S \operatorname{div} V \cdot u \, da \quad \forall V \in C_0^\infty(S, \mathbb{R}^3)$$

More general: $u \in L^2(S)$ has weak derivative $\partial^{(\beta)} u \in L^2(S)$, if

$$\int_S \partial^{(\beta)} u \cdot \ell \, da = (-1)^{|\beta|} \int_S u \, \partial^\beta \ell \, da \qquad \forall \ell \in C_0^\infty(S).$$

Def 3.3 (Sobolev spaces on a surface)

Let $S \subset \mathbb{R}^3$ regular surface, $m \geq 0$.

$$H^m(S) = \{ u \in L^2(S) : \exists\, \partial^{(\beta)} u \in L^2(S) \ \forall\, 0 \leq |\beta| \leq m \}$$

$$\|u\|_{m,S}^2 = \sum_{|\beta| \leq m} \int_S |\partial^{(\beta)} u|^2 \, da$$

If $\Gamma := \partial S \neq \emptyset$:

$$H_0^m(S) = \{ u \in H^m(S) : u|_\Gamma = 0 \}$$

The dual space of $H^m(S)$ is given by

$$H^{-m}(S) = \{ \ell : H^m(S) \to \mathbb{R} : \ell \text{ linear} \}$$

Remark:
- $H^m(S)$ is Hilbert space (complete, reflexive, ...)
- $H^m \subset L^2 \subset H^{-m}$
- Sobolev embedding: $H^2(S) \hookrightarrow C^{0,\alpha}(S)$, $H^1(S) \not\hookrightarrow C^{0,\alpha}(S)$ $[\alpha \in (0,1)]$

Dual pairing: $\langle x' | x \rangle := x'(x), \quad x' \in X', \; x \in X.$

Green's identity for vector field $v \in C^2(S, \mathbb{R}^3)$:

$$\int_S \mathrm{div}_S v \cdot \ell \, da = -\int_S g(v, \nabla_S \ell)\, da + \int_{\partial S} \ell \cdot g(v, n_S)\, ds \qquad \forall \ell \in C^1(S).$$

Def. 3.4 (Weak Laplace-Beltrami)

Let $S \subset \mathbb{R}^3$ regular, $u \in H^1(S)$. Then a weak Laplace-Beltrami operator $\Delta_S u \in H^{-1}(S)$ is defined by

$$\langle \Delta_S u | \ell \rangle := \int_S \Delta_S u \cdot \ell \, da := -\int_S g(\nabla_S u, \nabla_S \ell)\, da \qquad \forall \ell \in H_0^1(S)$$

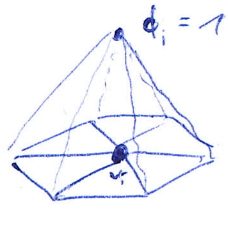
For $A \subset S$:

$$\int_A \Delta_S u \cdot \ell \, da := -\int_A g(\nabla_S u, \nabla_S \ell)\, da + \int_{\partial A} \ell \cdot g(\nabla_S u, n_S)\, ds \qquad \forall \ell \in H_0^1(S)$$

Linear FEM on a discrete surface

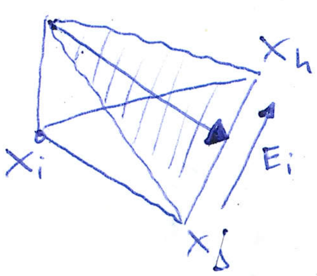
Linear FEM basis $(\phi_1, \ldots, \phi_n)$, $\phi_i : \mathcal{M}_h \to \mathbb{R}$,

$\phi_i(X_j) = \delta_{ij}$, $\quad \phi_i|_f$ affine for each $f \in \mathcal{F}$.



$\Rightarrow \phi_i \in C^0(\mathcal{M}_h)$ and $\phi_i \in H^1(\mathcal{M}_h)$.

Notation: $\tilde{\phi}_i := \phi_i \circ X : \Omega_h \to \mathbb{R}$



$$\nabla_h \phi_i|_f := \nabla_{\mathcal{M}_h} \phi_i|_f \approx \frac{1}{h_i} \cdot \frac{E_i^\perp}{\|E_i\|} = \frac{(X_k - X_j)^\perp}{2a_f}$$

Indeed: $DX|_f \, G_f^{-1} \nabla_{\mathbb{R}^2} (\phi_i \circ X_f) = \dfrac{(X_k - X_j)^\perp}{2a_f}$

where $\nabla_{\mathbb{R}^2}(\phi_i \circ X) \in \{(-1,-1)^T, (1,0)^T, (0,1)^T\}$

Since $\sum_i \tilde{\phi}_i(\xi) = 1 \quad \forall \xi \in \Omega_h$; we have on $f = (v_i, v_j, v_k)$

$$\nabla_h \phi_i|_f + \nabla_h \phi_j|_f + \nabla_h \phi_k|_f = 0.$$

A discrete function $u : \mathcal{M}_h \to \mathbb{R}$ is a pw. affine function, i.e. if $\bar{u} := (u_i)_i$, $u_i := u(X_i)$, $\tilde{u} := u \circ X : \Omega_h \to \mathbb{R}$:

$$\tilde{u}(\xi) = \sum_{i=1}^{n} u_i \tilde{\phi}_i(\xi).$$

Furthermore:
$$\nabla_h u|_f = \frac{u_j - u_i}{2a_f}(X_i - X_k)^\perp + \frac{u_k - u_i}{2a_f}(X_j - X_i)^\perp$$

$$H^1_0(S) = \begin{cases} \{u \in H^1(S) : u|_\Gamma = 0\}, & \Gamma = \partial S \neq \emptyset \\ \{u \in H^1(S) : \int_S u \, da = 0\}, & \partial S = \emptyset \end{cases}$$

Def. 3.4: Weak Laplace-Beltrami $\Delta_S : H^1_0(S) \to H^{-1}_0(S)$,

s.t. $\langle \Delta_S u \,|\, \ell \rangle := - \int_S g(\nabla_S u, \nabla_S \ell)\, da \quad \forall \ell \in H^1_0(S)$.

Def. 3.5: (Linear FEM space)

If M_h denotes a discrete surface with n vertices, we set

$$\mathcal{X}_h := \operatorname{span}\{\phi_1, \ldots, \phi_n\}, \quad \phi_i : M_h \to \mathbb{R}, \quad \phi_i(X_j) = \delta_{ij}$$

ϕ_i affine on each face of M_h.

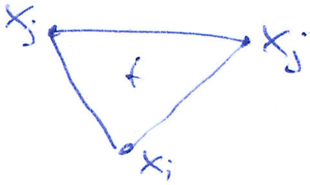
Remark: $\mathcal{X}_h \subset C^0(M_h)$, $\mathcal{X}_h \subset H^1(M_h)$.

Remark: Have omitted rigorous definition of Sobolev spaces on polyhedral meshes here.

Recall:

$$(*) \quad \nabla_h \phi_i|_f := \nabla_{M_h} \phi_i|_f := DX|_f \, G_f^{-1} \nabla_{\mathbb{R}^2}(\phi_i \circ X_f)$$
$$= (2a_f)^{-1} (X_k - X_j)^\perp$$

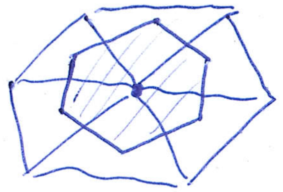
$$(**) \quad \nabla_h \phi_i|_f + \nabla_h \phi_j|_f + \nabla_h \phi_k|_f = 0$$



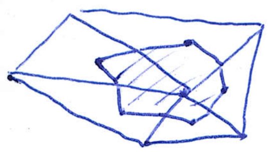
$$\Rightarrow \nabla_h u|_f = \frac{u_j - u_i}{2a_f}(X_i - X_k)^\perp + \frac{u_k - u_i}{2a_f}(X_j - X_i)^\perp$$

Discrete Laplace-Beltrami & cotan formula:

Let $v_i \in \mathcal{V}$ and $A_i \subset M_h$ associated with v_i.



barycentric cell



Voronoi cell

Important: ∂A_i has to intersect all adjacent edges at midpoint!

In the spirit of Def. 3.4 we set for $\phi_j \in H^1(M_h)$:

$$\int_{A_i} \Delta_h u \cdot \phi_j \, da := -\int_{A_i} \langle \nabla_h u, \phi_j \rangle \, da + \int_{\partial A_i} \langle \nabla_h u, N_s \rangle \phi_j \, ds$$

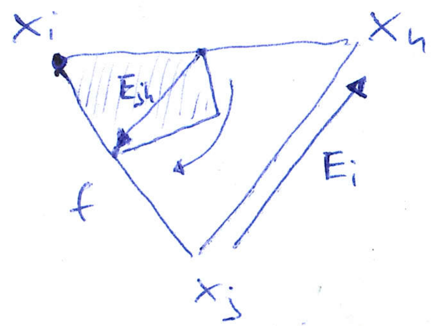
Summing over all j:

$$\int_{A_i} \Delta_h u \, da = \sum_{j=1}^{n} \int_{A_i} \Delta_h u \, \phi_j \, da$$

$$= \sum_{j=1}^{n} \sum_{f : v_i \in f} \left(-\int_{A_i \cap T(f)} \langle \nabla_h u, \nabla_h \phi_j \rangle \, da + \int_{\partial(A_i \cap T(f))} \phi_j \langle \nabla_h u, N_s \rangle \, ds \right)$$

$$= \sum_{f: v_i \in f} \int_{\partial A_i \cap T(f)} \langle \nabla_h u, N_s \rangle \, ds$$

Now consider one triangle $T(f) = (x_i, x_j, x_h)$



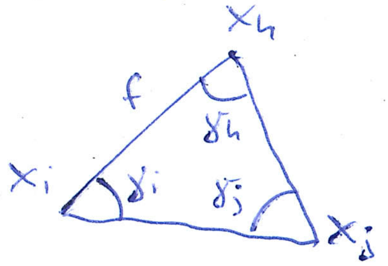
$$E_{jh} = -\tfrac{1}{2} E_i = \tfrac{1}{2}(x_j - x_h)$$

$$\int_{\partial A_i \cap T(f)} \langle \nabla_h u, N_s \rangle \, ds = - \int_{-E_{jh}} \langle \nabla_h u, N_s \rangle \, ds$$

$$= \| E_{jh} \| \cdot \left\langle \nabla_h u|_f , \frac{E_{jh}^{\perp}}{\| E_{jh} \|} \right\rangle$$

$$= (u_j - u_i) \frac{\langle (x_i - x_h), (x_j - x_h) \rangle}{4 a_f}$$

$$+ (u_h - u_i) \frac{\langle (x_j - x_i), (x_j - x_h) \rangle}{4 a_f}$$



$$a_f = \tfrac{1}{2} \sin \gamma_j \, \| x_j - x_h \| \cdot \| x_j - x_i \|$$

$$= \tfrac{1}{2} \sin \gamma_h \, \| x_h - x_i \| \cdot \| x_h - x_j \|$$

Hence:

$$\cos \gamma_j = \frac{\langle (x_j - x_i), (x_j - x_h) \rangle}{\| x_j - x_i \| \cdot \| x_j - x_h \|} = \sin \gamma_j \frac{\langle (x_j - x_i), (x_j - x_h) \rangle}{2 a_f}$$

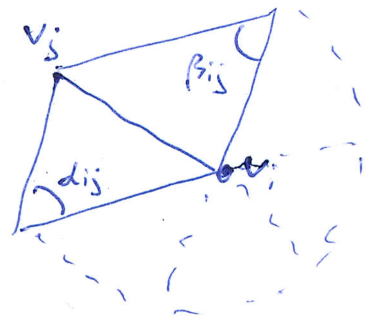
$$\cos \gamma_h = \;\text{— '' ——}\; = \sin \gamma_h \frac{\langle (x_h - x_i), (x_h - x_j) \rangle}{2 a_f}$$

Finally, due to $\cot\theta = \frac{\cos\theta}{\sin\theta}$:

$$\int_{\partial A_i \cap T} \langle \nabla_h u, N_S \rangle \, ds = \frac{1}{2} \cdot \left((u_j - u_i) \cot\gamma_j + (u_k - u_i) \cot\gamma_k \right)$$

Summing over all f, $v_i \in f$:

$$(*) \quad \int_{A_i} \Delta_h u \, da = \frac{1}{2} \sum_{v_j \in N_1(v_i)} \left(\cot\alpha_{ij} + \cot\beta_{ij} \right) (u_j - u_i)$$



Using the paradigm of spatial averaging:

$$(\Delta_h u)_i := (\Delta_h u)(X_i)$$

$$(**) \quad := \frac{1}{|A_i|} \int_{A_i} \Delta_h u \, da = \frac{1}{2|A_i|} \sum_{v_j \in N(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})(u_j - u_i)$$

<u>Note</u>: (*) does <u>not</u> depend on specific choice of A_i

(**) does depend —— " —————— !

Discrete Laplace-Beltrami from a FEM point of view

Stiffness matrix $L \in \mathbb{R}^{n,n}$:

$$L_{ij} = \int_{M_h} g(\nabla_h \phi_i, \nabla_h \phi_j)\, da$$

Define $\Delta_h : \mathcal{X}_h \to (\mathcal{X}_h)'$ via

$$\langle \Delta_h u | v \rangle := - \int_{M_h} g(\nabla_h u, \nabla_h v)\, da$$

$$= - \sum_{i,j=1}^{n} u_i v_j \int_{M} g(\nabla_h \phi_i, \nabla_h \phi_j)\, da$$

$$= - \sum_{i,j=1}^{n} L_{ij} u_i v_j$$

for $u(x) = \sum_i u_i \phi_i(x)$, $v(x) = \sum_j v_j \phi_j(x)$.

Direct verification:

$$L_{ij} = -\frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij}) \quad , \ i \neq j$$

Finally:

$$L_{ii} = \sum_{f \in \mathcal{F}} \int_{T(f)} \langle \nabla_h \phi_i, \nabla_h \phi_i \rangle\, da \overset{(**)}{=} -\sum_{f \in \mathcal{F}} \int_{T(f)} \langle \nabla_h \phi_i, \nabla_h \phi_j + \nabla_h \phi_k \rangle\, da$$

$$= - \sum_{i \neq j} L_{ij}$$

$$\Rightarrow \langle \Delta_h u | \phi_i \rangle = - \sum_{k=1}^{n} L_{ik} u_k = \frac{1}{2} \sum_{v_j \in N(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij})(u_j - u_i)$$

which corresponds to $\int_{A_i} \Delta_h u \, da$ derived above.

Define (lumped) mass matrix $M = \mathrm{diag}(m_1, \ldots, m_n)$, with

$$m_i = \int_{M_h} \phi_i \, da,$$

get pointwise evaluation by spatial averaging:

$$(\Delta_h u)_i = -(M^{-1} L \bar{u})_i \quad , \ \bar{u} = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$$

Remark: If $A_i = \bigcup_{f: i \in f} T(f)$ then $m_i = \frac{1}{3}|A_i|$,
i.e. m_i coincides with the area of the barycentric cell associated with v_i.

Bottom line:

$$\langle \Delta_h u | \phi_i \rangle \;\hat{=}\; \int_{A_i} \Delta_h u \, da$$ (independent of A_i, midpoint condition!)

$$(\Delta_h u)_i \;\neq\; \frac{1}{|A_i|} \int_{A_i} \Delta_h u \, da$$ ($\hat{=}$ if A_i barycentric cell)

See "Meyer, Desbrun, Schröder, Barr, 2002"

Mean curvature functional & function

Using $\Delta_S id = -H_p n(p)$ we define $\vec{H}_h \in (\mathcal{X}_h^1)^3$ via

$$\langle \vec{H}_h | \phi_i \rangle := -\int_{M_h} \Delta_h id \, \phi_i \, da = -\frac{1}{2} \sum_{v_j \in N(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})(x_j - x_i)$$

Spatial averaging leads to pointwise mean curvature vector:

$$(\vec{H}_h)_i := \sum_{v_j \in N(v_i)} M^{ij} \langle \vec{H}_h | \phi_j \rangle = -\frac{1}{2m_i} \sum_{j \sim i} (\cot\alpha_{ij} + \cot\beta_{ij})(x_j - x_i)$$

Different Laplace-Beltrami used in GP:

$$(Lu)_i \approx \langle \Delta_h u | \ell_i \rangle = \int_{M_h} \Delta_h u \cdot \ell_i \, da$$

for some function $\ell_i : M_h \to \mathbb{R}$ associated with v_i, i.e. local support in the vicinity of v_i.

General setup: $(Lu_0)_i = \sum_j w_{ij}(u_j - u_i)$

Umbrella operator:
~~Umbrella operator~~ $w_{ij} = 1$ if $i \sim j$ (Taubin, 95)

~~Tutte Laplacian $w_{ij} = 1$ / #neighbours of v_i $\cot\beta_{ij}$~~

"See Wardetzky, Mathur, Kälberer, Grinspun, 2007"

3.3 Convergence of discrete mean curvature

Based on

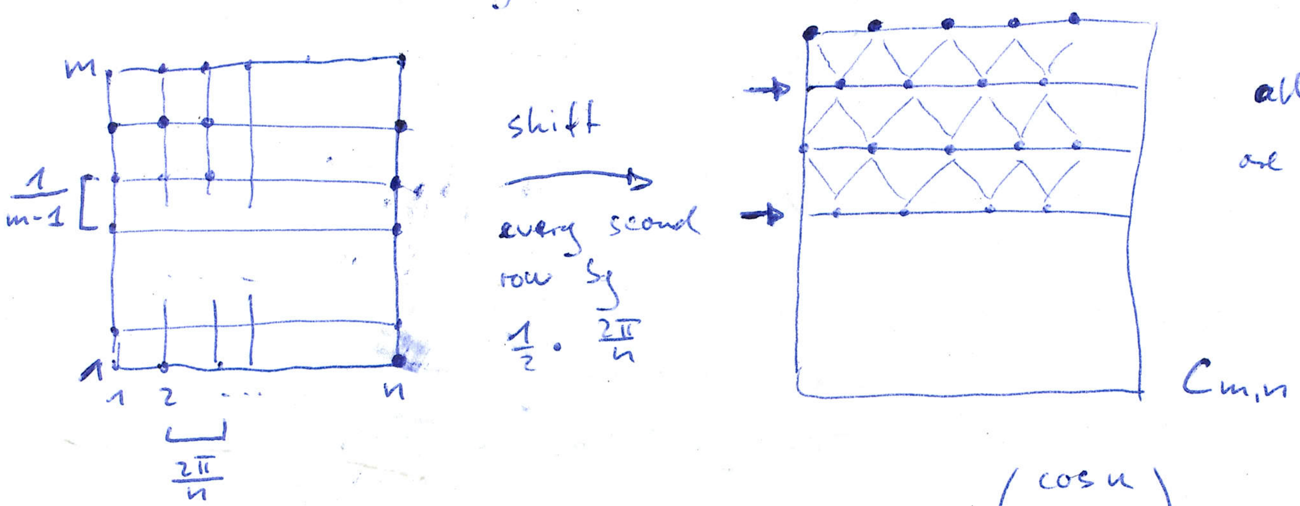
- Wardetzky, 2006, PhD thesis
- Hildebrandt, Polthier, Wardetzky, 2006
- Wardetzky, 2008, MFO report

Consider a sequence of discrete surfaces M_n converging (in Hausdorff measure) to a smooth embedded surface S.

Q: What are conditions s.t. (discrete) geometric objects on M_n converge to corresponding objects on S?

Example (Schwarz lantern)

Discretization of cylinder with unit height & unit radius.



all triangles are congruent!

$$x : [0,2\pi) \times [0,1] \to \mathbb{R}^3, \quad (u,v) \mapsto \begin{pmatrix} \cos u \\ \sin u \\ v \end{pmatrix}$$

Note that $C_{m,n} \to C$ but:

The triangle normals of $C_{m,n}$ approach normals of C iff. $\frac{m}{n^2} \to 0$

$\Rightarrow$ one cannot expect convergence of geometric properties from pointwise convergence alone!

Two ~~of~~ main results of Wardetzky et.al:

(1) If $M_h \to S$ in Hausdorff distance, then convergence of normals is equivalent to convergence of metric tensors, area, and Laplace-Beltrami op.

(2) To prove convergence of geometric properties it often suffices to postulate pointwise convergence $\oplus$ normal convergence.

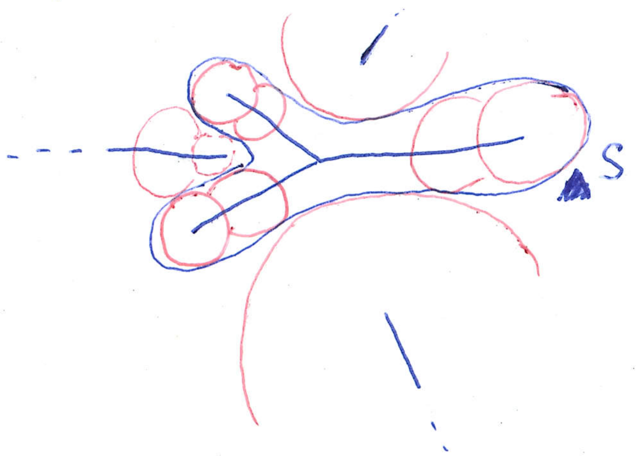
But: Convergence is shown in weak/integrated/distributional sense, pointwise conv. of geom. properties (e.g. curvature) cannot be expected.

Shortest distance map & geometric splitting

Let $S \subset \mathbb{R}^3$ regular surface, M_h sequence of meshes $(h \to 0)$ approximating S. Convergence analysis needs map between M_h and S to compare objects!

Def. 3.6 (Medial axis & reach)

Let $A \subset \mathbb{R}^3$ closed subset. The medial axis of A is the set of those points in $\mathbb{R}^3$ which do not have a unique closest neighbor in A. The reach of A is the distance of A to its medial axis.



medial axis of S $\hat{=}$ locus of centers of spheres tangentially touching S

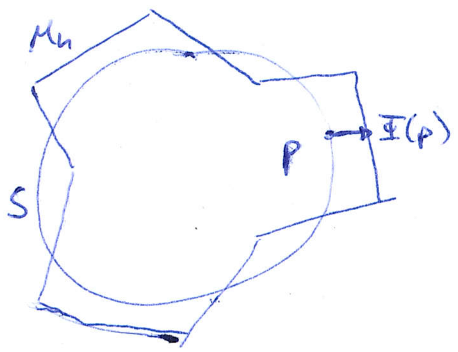
$$\text{reach}(S) \leq \inf_{p \in S} \frac{1}{\max(|\kappa_1(p)|, |\kappa_2(p)|)}$$

$$\text{reach}(M_h) = 0$$

Def. 3.7 (Normal graph, shortest distance map)

M_h is <u>normal graph</u> over S if it is within the reach of S and the map which maps each point on M_h to its closed point on S is a bijection $\Psi : M_h \to S$.

The inverse $\Phi = \Psi^{-1} : S \to M_h$ is <u>shortest distance map</u>.



M_h not in the reach of S!

$\Psi : M_h \to S$ is not a bijection!

Φ induces a metric on S which allows one to compose M_h and S as metric spaces:

$$(u, v) \longmapsto \langle d\Phi(u), d\Phi(v) \rangle_{\mathbb{R}^3} \quad \text{a.e.}$$

u, v tangent vectors on S

Def. 3.8 (Metric distortion tensor)

There exists a symmetric, positive definite 2×2 matrix field on S which represents a linear mapping $A(p) : T_p S \to T_p S$, such that for a.e. $p \in S$:

$$g_A(u, v) := \langle A(p) u, v \rangle_{\mathbb{R}^3} = \langle d\Phi(u), d\Phi(v) \rangle_{\mathbb{R}^3}$$

for $u, v \in T_p S$. A is smooth on the pre-image of the interior of triangles and denoted by <u>metric distortion tensor</u>.

The next theorem shows that A only depends on

(i) distance between S and M_h (ii) angle between normals

(iii) curvature of smooth surface

<u>Thm 3.9</u> (Geometric splitting, Wardetzky '06)

Let M_h be a closed ~~polyhedral~~ discrete surface with face normals N, which is normal graph over an embedded, closed, smooth surface S with normal field n. Then

$$A = P \circ Q^{-1} \circ P \qquad \text{a.e. on } S$$

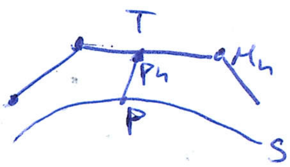
The linear operator P, and Q can be represented by spd. matrices which diagonalize as

$$P \sim \begin{pmatrix} 1+\phi\kappa_1 & \\ & 1+\phi\kappa_2 \end{pmatrix}, \qquad Q \sim \begin{pmatrix} \langle n, N\circ\Phi\rangle^2 & \\ & 1 \end{pmatrix}$$

where κ_1, κ_2 are principle curvatures on S, $\phi : S \to \mathbb{R}$ the (signed) distance, i.e. $|\phi(p)| = \|p - \Phi(p)\|_{\mathbb{R}^3}$.

<u>Remark</u>: P is pos. def. since M_h is in the reach of S, i.e. $\phi <$ reach(S) and hence $1 + \phi \cdot \kappa_i > 0$.

<u>Proof</u>: Consider one fixed triangle $T \in M_h$, let $\Phi : T \to S$.



$$\Phi(p) = p + \phi(p)\, n(p)$$

$$\Longrightarrow \quad \Psi(p_h) = p_h - (\phi\circ\Psi)(p_h) \cdot (n\circ\Psi)(p_h)$$

The differential $d_{p_h}\Psi : T_{p_h}M_h \to T_pS$, $p = \Psi(p_h)$, is given for $V \in T_{p_h}M_h$ by:

$$d_{p_h}\Psi(V) = V - (n\circ\Psi)\cdot\Big(d_p\phi\big(d_{p_h}\Psi(V)\big)\Big)$$

$$- (\phi\circ\Psi)\cdot\Big(d_p n\big(d_{p_h}\Psi(V)\big)\Big) \qquad (*)$$

where $d_p\phi$ is a linear form on T_pS.

Since $\tilde{V} := d_{p_h}\Psi(V) \in T_pS$ we get:

$$\begin{aligned} 0 &= \langle d_{p_h}\Psi(V), n\circ\Psi \rangle \\ &= \langle V, n\circ\Psi\rangle - d_p\phi(\tilde{V})\cdot \underbrace{\langle n\circ\Psi, n\circ\Psi\rangle}_{=1} \\ &\qquad - (\phi\circ\Psi)\cdot \underbrace{\langle d_pn(\tilde{V}), n\circ\Psi\rangle}_{=0} \end{aligned}$$

$$\Rightarrow \quad d_p\phi(\tilde{V}) = \langle V, n\circ\Psi\rangle$$

plug this into $(*)$; use $d_pn = S_p$:

$$\begin{aligned} d_{p_h}\Psi(V) = V &- (n\circ\Psi)\cdot\langle V, n\circ\Psi\rangle \\ &- (\phi\circ\Psi)\cdot S_p(d_{p_h}\Psi(V)) \end{aligned}$$

$$\Rightarrow \quad \underbrace{\left(\mathrm{Id} + (\phi\circ\Psi)S_p\right)}_{=:\,P\,:\,T_pS\to T_pS} d_{p_h}\Psi(V) = \underbrace{V - \langle V, n\circ\Psi\rangle\, n\circ\Psi}_{=:\,\tilde{Q}\,:\,T_{p_h}M_h\to T_pS}$$

where P is symmetric endomorphism on T_pS, $\tilde{Q}$ projection from $T_{p_h}M_h$ onto T_pS.

$$P\, d_{p_h}\Psi(V) = \tilde{Q}(V) \quad \Rightarrow \quad d_{p_h}\Psi = P^{-1}\circ\tilde{Q}$$

$$\Rightarrow \quad d\Phi = d\Psi^{-1} = \tilde{Q}^{-1}\circ P$$

Define a linear mapping $Q^{-1} : T_pS \to T_pS$ via

$$\langle Q^{-1}(W_1), W_2\rangle = \langle \tilde{Q}^{-1}(W_1), \tilde{Q}^{-1}(W_2)\rangle \quad \forall W_{1,2}$$

Then we get:

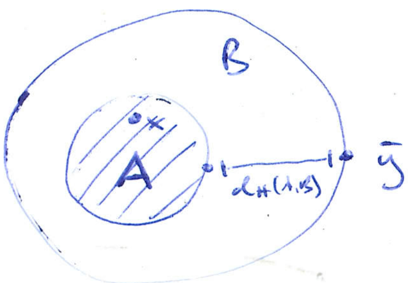
$$\begin{aligned}\langle A(W_1), W_2\rangle &= \langle d\Phi(W_1), d\Phi(W_2)\rangle \\ &= \langle (\tilde{Q}^{-1}\circ P)(W_1), (\tilde{Q}^{-1}\circ P)(W_2)\rangle \\ &= \langle (\tilde{Q}^{-1}\circ P)(W_1), P(W_2)\rangle \\ &= \langle (P\circ \tilde{Q}^{-1}\circ P)(W_1), W_2\rangle\end{aligned}$$

Furthermore, S_p has eigenvalues κ_1, κ_2 (and 0) and $\tilde{Q} : N^\perp \to n^\perp$ has eigenvalues $\langle n, N\circ\Phi\rangle$ and 1 (and 0). $\square$

Def. 3.10 (Hausdorff distance)

Let $A, B \subset \mathbb{R}^d$ be non-empty subsets. Then the Hausdorff distance between A and B is defined as

$$d_H(A,B) := \max\left\{ \sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}$$



$A\subset B$

$\forall x\in A:\ \inf_{y\in B} d(x,y) = 0$

but $\inf_{x\in A} d(x,\bar{y}) > 0$

Notation: In the following we often consider $(M_{h_n})_n \xrightarrow{\mathcal{H}} S$, i.e. $h_n \to 0$ if $n\to\infty$ and $d_H(M_{h_n}, S) \to 0$ if $n\to\infty$.

Write $M_n := M_{h_n}$, where e.g. $n = \#$ nodes of M_n.

Def. 3.11 (Totally normal convergence)
Let $S \subset \mathbb{R}^3$ regular surface, $(M_n)_n$ sequence of discrete surfaces with shortest distance maps $\Phi_n : S \to M_n$ and normal fields $(N_n)_n$. Then $(M_n)_n$ is said to converge ~~totally~~ normally to S if $\| N_n \circ \Phi_n - n \|_\infty \to 0$.
Normal convergence is said to be totally normal if additionally $d_H(M_n, S) \to 0$.

Remark: $\forall n \in \mathbb{N}$: M_n induces metric on S by A_n.
For a.e. $p \in S$: $A_n(p) : T_pS \to T_pS$ is endomorphism.
Set $\| A_n \|_\infty := \operatorname{ess\,sup}_{p \in S} \| A_n(p) \|$
Convergence of metric tensors means $\| A_n - \mathbb{1} \|_\infty \to 0$.

Theorem 3.12 (Equivalent conditions for convergence, Ward. '06)
Let $S \subset \mathbb{R}^3$ compact & regular surface, let $(M_n)_n$ sequence that are normal graphs over S with $M_n \xrightarrow{\mathcal{H}} S$.
Then:
$$\| N_n \circ \Phi_n - n \|_\infty \to 0 \iff \| A_n - \mathbb{1} \|_\infty \to 0$$

Proof: Let A_n sequence of metric dist. tensors induced by M_n, then $A_n = P_n \circ Q_n^{-1} \circ P_n$ (cf. Thm. 3.9).
$M_n \xrightarrow{\mathcal{H}} S \implies \phi_n \to 0 \implies \| P_n - \mathbb{1} \|_\infty \to 0$
and normal convergence in L^∞ is equivalent to $\langle n, N_n \circ \Phi_n \rangle \to 1$
$\square$

Convergence of mean curvature functional

For $u: S \to \mathbb{R}$ define $\nabla_A u$ based on metric g_A: (cf. Def. 3.8)

$$\forall V \in TS: \quad g_A(\nabla_A u, V) := \frac{d}{dt} u(\cdot + tV)\Big|_{t=0} = \langle \nabla_S u, V \rangle_{\mathbb{R}^3}$$

$$\Rightarrow \nabla_A u = A^{-1} \nabla_S u$$

Next: define two weak Laplace-Beltrami operators, $\Delta_S, \Delta_n : H^1_0(S) \to H^{-1}_0(S)$, one corresponds to Def. 3.4, the other one is induced by Φ resp. g_A:

$$\langle \Delta_S u | v \rangle := - \int_S g(\nabla_S u, \nabla_S v)\, da = - \int_S \langle \nabla_S u, \nabla_S v \rangle_{\mathbb{R}^3}\, da$$

$$\langle \Delta_n u | v \rangle := - \int_S g_{A_n}(\nabla_{A_n} u, \nabla_{A_n} v) \sqrt{\det A_n}\, da$$

$$= - \int_S \langle A_n^{-1} \nabla_S u, \nabla_S v \rangle_{\mathbb{R}^3} \sqrt{\det A_n}\, da$$

Remark: Can define Lapl.-Belt.-op. directly intrinsically on M_n, but it is difficult to construct Sobolev spaces $H^1_0(M_n)$. However, weak differentiability is preserved under bi-Lipschitz map (cf. Rademacher's theorem), hence one can identify $H^1_0(M_n)$ with $H^1_0(S)$ via Φ_n. Finally, one can show that Δ_n equals the pullback of the intrinsically defined Laplace-Beltrami op. on M_n.

Thm 3.13 (Lemma 3.2.2 in Word. '06)

Let M_n be normal graph over S, $\Phi: S \to M_n$ shortest dist. map.

(i) $u \in L^2(\Phi(S)) \iff u \circ \Phi \in L^2(S)$

(ii) $u \in H^1_0(\Phi(S)) \iff u \circ \Phi \in H^1_0(S)$

Recall: Thm 2.11: $\Delta_S \, id = -H_p \, n(p)$.

Def. 3.14: (Discrete mean curvature functional / Weak mean curv.)

Let $id_S : S \to \mathbb{R}^3$, $id_{M_n} : M_n \to \mathbb{R}^3$ embeddings, set

$$id_n^* := id_{M_n} \circ \Phi : S \to \mathbb{R}^3.$$

Then define a (discrete) weak mean curvature functional by

$$\vec{H} = -\Delta_S \, id_S \quad \in (H_0^{-1}(S))^3$$

$$\vec{H}_n = -\Delta_n \, id_n^* \quad \in (H_0^{-1}(S))^3$$

Thm. 3.15 (Connection with cotan formula)

$$\langle \vec{H}_n \mid \phi_i \rangle = -\frac{1}{2} \sum_{v_j \in N(v_i)} (\cot \alpha_{ij} + \cot \beta_{ij})(X_j - X_i)$$

where ϕ_i is nodal basis function at v_i.

Notation: For $F \in (H_0^{-1}(S))^d$, the operator norm is defined as

$$\| F \|_{H_0^{-1}(S)} := \sup_{\substack{u \in H_0^1(S) \\ u \neq 0}} \frac{\| \langle F \mid u \rangle \|_{\mathbb{R}^d}}{\| u \|_{H_0^1(S)}}$$

Theorem 3.16 (Convergence of weak mean curv., Ward. '06)

Let M_n be normal graph over regular surface $S \subset \mathbb{R}^3$.

$$\| \vec{H} - \vec{H}_n \|_{H_0^{-1}(S)} \leq \sqrt{|S|} \cdot \left[\| \sqrt{\det A_n} \, A_n^{-1} - \mathbb{1} \|_\infty + \| \sqrt{\det A_n} \, A_n^{-1} \|_\infty \cdot \| Id - d\Phi_n \|_\infty \right]$$

where $\| Id - d\Phi_n \|_\infty := \operatorname{ess\,sup}_{p \in S} \| (Id - d\Phi_n)_p \|$, and $Id - d\Phi_n : T_p S \to \mathbb{R}^3$. In particular, if $(M_n)_n$ converges totally normal to S, then $\vec{H}_n \to \vec{H}$ in $H_0^{-1}(S)$.

Proof: Triangle-inequality:

$$\vec{H} - \vec{H}_n = (\Delta_S id - \Delta_n id) + (\Delta_n id - \Delta_n id_n^*)$$

Since $\langle \nabla_S id, V \rangle = V$ and $\langle \nabla_S (id_M \circ \Phi), V \rangle = d\Phi(V)$ for any vector field V on S, we get with Hölder's inequality

$$\| \langle \Delta_S id - \Delta_n id \,|\, u \rangle \|_{\mathbb{R}^3} = \Big\| \int_S \big(\sqrt{\det A_n}\, A_n^{-1} - \mathbb{1}\big) \nabla_S u \, da \Big\|_{\mathbb{R}^3}$$

$$\leq \sqrt{|S|}\, \| \sqrt{\det A_n}\, A_n^{-1} - \mathbb{1} \|_\infty \, \|u\|_{H_0^1(S)}$$

$$\| \langle \Delta_n id - \Delta_n id_n^* \,|\, u \rangle \|_{\mathbb{R}^3} = \Big\| \int_S \langle \nabla_S id - \nabla_S id_n^*, \sqrt{\det A_n}\, A_n^{-1} \nabla_S u \rangle \, da \Big\|_{\mathbb{R}^3}$$

$$\leq \Big\| \int_S (Id - d\Phi_n) \big(\sqrt{\det A_n}\, A_n^{-1}\big) \nabla_S u \, da \Big\|_{\mathbb{R}^3}$$

$$\leq \sqrt{|S|}\, \| \sqrt{\det A_n}\, A_n^{-1} \|_\infty \, \| Id - d\Phi_n \|_\infty \, \|u\|_{H_0^1(S)}$$

Under totally normal convergence we get $\| A_n - \mathbb{1} \|_\infty \to 0$, hence $\| \sqrt{\det A_n}\, A_n^{-1} - \mathbb{1} \|_\infty \to 0$.

Remains to show: $\| Id - d\Phi_n \|_\infty \to 0$.

Consider a triangle $T_n \in M_n$, let $n \circ \Phi^{-1}$ be the pull-back of normal field n on S to T_n. From the proof of Thm 3.9 we know $d\Phi = \tilde{Q}^{-1} \circ P$ with $\tilde{Q}(V) = V - \langle n \circ \Phi^{-1}, V \rangle (n \circ \Phi^{-1})$. Totally normal convergence implies $P \to Id$ & $\tilde{Q} \to Id$, hence $d\Phi \to Id$ almost everywhere. $\square$

Counterexample of L^2-convergence (Ward. '06)

Definition 3.17 (Discrete mean curv. function)

Let $(\phi_i)_i$ hat basis of linear FEM on M_h.

$$\vec{H}_h(p) := \sum_{i,j=1}^{n} \langle \vec{H}_h | \phi_i \rangle M^{ij} \phi_j(p)$$

where $M_{ij} = \int_{M_h} \phi_i \phi_j \, da$ resp. $M_{ij} = \int_{M_h} \phi_i \, da \cdot \delta_{ij}$,

and $M_{ik} M^{kj} = \delta_{ij}$, i.e. $(M^{ij})_{ij} = M^{-1}$.

Now denote by $\vec{H}(p) = H_p n(p)$ the smooth MC vector of S, let $(\vec{H}_n)_n$ with $\vec{H}_n := \vec{H}_{h_n}$ discrete mean curv. functions associated with sequence $(M_n)_n$, $M_n = M_{h_n}$.

We show: $\| \vec{H} - \vec{H}_n \|_{L^2} \not\to 0$ in general.

Consider cylinder of height 2π and radius 1.

Vertices of M_n:

$$u = \frac{i\pi}{n}, \quad i = 0, \dots, 2n-1$$

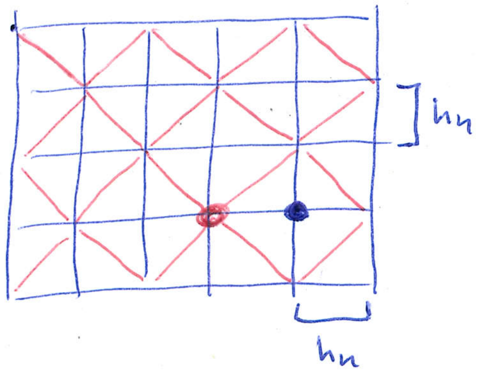
$$v = \begin{cases} 2j \sin \frac{\pi}{2n}, & j < 2n \\ 2\pi, & j = 2n \end{cases}$$

$$(u,v) \mapsto \begin{pmatrix} \cos u \\ \sin u \\ v \end{pmatrix}$$

$\Rightarrow$ regular quad-grid of edge length $h_n = 2 \sin \frac{\pi}{2n}$.

It will depend on tesselation whether L^2-conv. or not!

Consider regular 4-8-tessellation (criss-cross)



Let $\mathcal{V}_h := \{ v \in \mathcal{V} : \text{valence of } v \text{ is } h \}$, $h \in \{4, 8\}$.

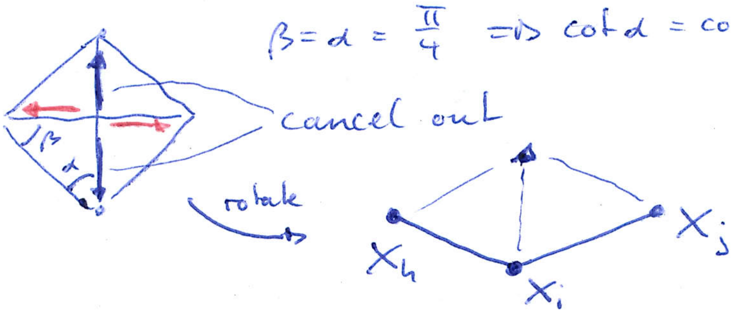
One can verify:

$$\langle \vec{H}_n \,|\, \phi_v \rangle = \lambda_n \cdot N_v \,, \quad \lambda_n = 2\left(1 - \cos\frac{\pi}{n}\right)$$

for $v \in \mathcal{V}_4$ <u>and</u> $v \in \mathcal{V}_8$; N_v outward vertex normal at v.

<u>$v \in \mathcal{V}_4$</u>:

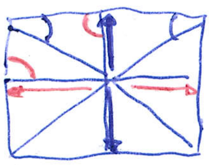
$\beta = \alpha = \frac{\pi}{4} \Rightarrow \cot\alpha = \cot\beta = 1$



$$\Rightarrow \langle H_n \,|\, \phi_v \rangle = -\frac{1}{2} \cdot \Big((X_j - X_i) + (X_h - X_i) \Big)$$

$$= \lambda_n \cdot N_v$$

<u>$v \in \mathcal{V}_8$</u>:



$\beta = \alpha = \frac{\pi}{2} \Rightarrow \cot\alpha = \cot\beta = 0$

$\beta = \alpha = \frac{\pi}{4} \Rightarrow$ see above.

Using Def. 3.17 with lumped mass matrix:

$$\vec{H}_n(p) = \sum_{v \in \mathcal{V}_4} m_v^{-1} \lambda_n \phi_v(p) N_v(p) + \sum_{v \in \mathcal{V}_8} m_v^{-1} \lambda_n \phi_v(p) N_v(p)$$

where

$$m_v = \begin{cases} \frac{2}{3} h_n^{-2} & , v \in \mathcal{V}_4 \\ \frac{4}{3} h_n^{-2} & , v \in \mathcal{V}_8 \end{cases}$$

Since $\frac{\lambda_n}{h_n^2} = 1 \ \forall n \in \mathbb{N}$: $\quad \vec{H}_n(p_4) = 2 \cdot \vec{H}_n(p_8)$

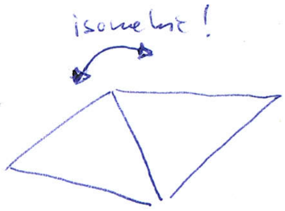
(such $\vec{H}_n(p) \geq c > 0 \ \forall p$)

Hence $\vec{H}_n$ is pw. affine function with oscillating values at $v \in \mathcal{V}_4$ resp. $v \in \mathcal{V}_8$ with ever growing frequencies as $n \to \infty$.

$\Rightarrow$ no convergence in L^2 !

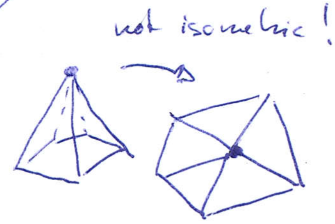
3.4 Notions of discrete curvature measures

<u>In applications</u>: isometric deformations $\Rightarrow$ extrinsic curvature measures!

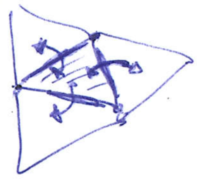


Gauss' Theorema Egregium:

- mean curv. concentrated at edges
- Gauss curv. concentrated at vertices



$\Rightarrow$ shape operator naturally associated with edges, however, discrete shape operators are usually associated with triangles by taking into account the bending accross the three edges



<u>Normals</u>:

- face normal, $T = T(f) = (x_0, x_1, x_2)$

$$N(f) = N_f = \frac{(x_1 - x_0) \times (x_2 - x_0)}{\| (x_1 - x_0) \times (x_2 - x_0) \|}$$

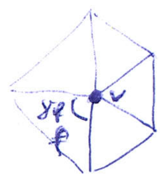
- edge normals, at edge midpoints

$$E = T(f_\ell) \cap T(f_r)$$

$$\Rightarrow N_E = \frac{\alpha_\ell N(f_\ell) + \alpha_r N(f_r)}{\| \alpha_\ell N(f_\ell) + \alpha_r N(f_r) \|}$$

with weigths $\alpha_\ell, \alpha_r \in \mathbb{R}$, e.g. $\alpha_r = \alpha_\ell = \frac{1}{2}$.

- vertex normals at $v \in \mathcal{V}$:



$$N_v = \frac{\sum_{f \in N(v)} \alpha_f N_f}{\| \sum_{f \in N(v)} \alpha_f N_f \|}, \quad \alpha_f \in \mathbb{R} \text{ weights.}$$

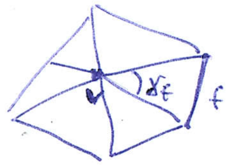
Different choices for α_f, e.g. $\alpha_f = 1$, $\alpha_f = a_f$, $\alpha_f = \gamma_f$.

Vertex-based curvature measures

Let $(A_v)_{v\in\mathcal{V}}$ with $\bigcup_v A_v = M_h$, $A_v \cap A_{v'} = \emptyset$, $v \in A_v$.

Define discrete Gauss-curvature (weakly!), s.t. a discrete version of Gauss-Bonnet holds:

$$\int_{A_v} K_h \, da := 2\pi - \sum_{f : v\in f} \gamma_f \qquad \text{(angle defect)}$$



$$\int_{M_h} K_h \, da = \sum_{v\in\mathcal{V}} \int_{A_v} K_h \, da = 2\pi|\mathcal{V}| - \underbrace{\sum_{v\in\mathcal{V}} \sum_{f : v\in f} \gamma_f}_{= \pi \cdot |\mathcal{F}|}$$

$$= 2\pi\left(|\mathcal{V}| - \tfrac{1}{2}|\mathcal{F}|\right) = 2\pi\left(|\mathcal{V}| + |\mathcal{F}| - |\mathcal{E}|\right),$$

as $3|\mathcal{F}| = 2|\mathcal{E}|$

Spatial averaging: $\quad K_h(v) := \dfrac{\int_{A_v} K_h \, da}{|A_v|}$

Vertex-based mean curvature (weak & pointwise) via discrete Laplace-Beltrami as in Sec. 3.2 & 3.3

$$\langle \vec{H}_h \mid \phi_i \rangle := -\int_{M_h} \Delta_h \, \mathrm{id} \cdot \phi_i \, da = -\frac{1}{2} \sum_{v_j \in N(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})(X_j - X_i)$$

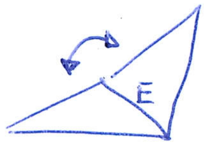
Spatial averaging:

$$(\vec{H}_h)_{v_i} := -\frac{1}{2m_i} \sum_{v_j \in N(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij})(X_j - X_i)$$

with m_i associated area weight with v_i, e.g. $m_i = \int_{M_h} \phi_i \, da$.

Edge-based curvature measures

Bending across edge is captured by dihedral angle Θ_E, $E = T(f_r) \cap T(f_\ell)$



$$\Theta_E := \sphericalangle\left(N(f_\ell) \times E ,\ E \times N(f_r) \right)$$

$(\Theta_E = \pi \text{ if } N(f_\ell) \parallel N(f_r))$

Intuitively: $\quad H_E \sim -2 \cos \frac{\Theta_E}{2}$ (*)

$(H_E > 0 \text{ if } \Theta_E > \pi,\ H_E < 0 \text{ if } \Theta_E < \pi)$

Taylor expansion about π leads to:

$$H_E \sim (\Theta_E - \pi) + O(|\Theta_E - \pi|^3) \quad (**)$$

Edge-based shape op.:

$$S_E = \frac{1}{3} H_E (N_E \times E) \otimes (N_E \times E),$$

where $a \otimes b = ab^T \in \mathbb{R}^{d,d}$ if $a, b \in \mathbb{R}^d$.

Note that $S_E E = 0$, $S_E N_E = 0$, $\operatorname{tr} S_E = H_E$.

Now: Formal derivation of (*) resp. (**).

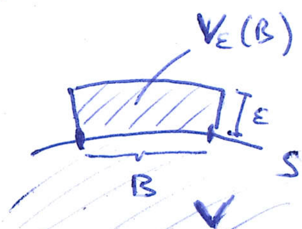
However, start with weak notion!

Convergence results by Cohen-Steiner & Morvan '03:

Let $V \subset \mathbb{R}^3$ convex body s.t. $S := \partial V$ is smooth & compact surface.

For $\varepsilon > 0$ and some area $B \subset S$ define the offset

$$V_\varepsilon(B) := \{ x + \varepsilon t n(x) : x \in B,\ t \in [0,1] \}$$



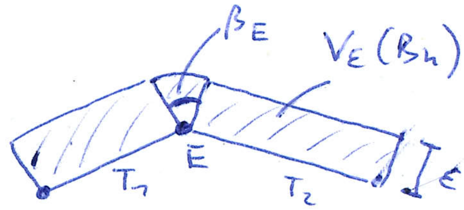
Steiner's tube formula

$$|V_\varepsilon(B)| = \varepsilon \int_B da + \frac{1}{2}\varepsilon^2 \int_B H\, da + \frac{1}{3}\varepsilon^3 \int_B K\, da$$

$$\Rightarrow \int_B H\, da = 2\varepsilon^{-2}\left(|V_\varepsilon(B)| - \varepsilon|B|\right) + O(\varepsilon). \qquad (***)$$

Now let $B_h \subset M_h$, where $\Theta_E > \pi \ \forall E \in M_h$ since $M_h \approx \partial V$ and V convex.

(i) Let $B_h = T_1 \cup T_2$, $E = T_1 \cap T_2$.



$$|V_\varepsilon(B)| - \varepsilon|B_h| = \frac{\beta_E}{2\pi}\cdot\varepsilon^2\pi\cdot\|E\|$$

where $\beta_E = \Theta_E - \pi$.

$$(***) \rightarrow \int_{B_h} H_h\, da := (\Theta_E - \pi)\cdot\|E\|.$$

(ii) arbitrary $B_h \subset T_1 \cup T_2$: $\int_{B_h} H_h\, da = (\Theta_E - \pi)\,\|E \cap B_h\|$

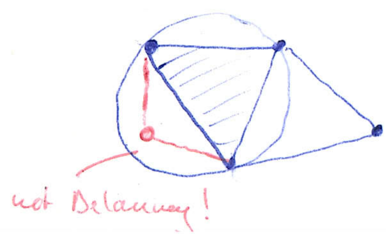
(iii) arbitrary $B_h \subset M_h$

$$\int_{B_h} H_h\, da = \sum_{E \subset B_h} (\Theta_E - \pi)\,\|E \cap B_h\|$$

since cone-like volumes sitting at a vertex $\sim \varepsilon^3$.

Def. 3.18: (Delaunay triangulation)

A delaunay triangulation of a set of points in $\mathbb{R}^2$ is a triangulation s.t. no point is inside the circumcircle of any triangle.



Definition 3.19 (ε-sample)

A point set $P \subset \mathbb{R}^3$ is an ε-sample of a regular surface $S \subset \mathbb{R}^3$ if for all $p \in S$: $$B_{r(\varepsilon,p)}(p) \cap P \neq \emptyset,$$

where $r(\varepsilon, p) := \varepsilon \cdot \operatorname{dist}(p, \operatorname{med}(S))$.

Theorem 3.20 (Convergence of weak edge-based curvature, CM '03)

Let $S \subset \mathbb{R}^3$ regular surface, $M_h = (\mathcal{V}, \mathcal{F}, \mathcal{E})$ approximating mesh and $\varepsilon > 0$ sufficiently small. If the set $\{E(v) : v \in \mathcal{V}\}$ is an ε-sample of S and the triangulation of M_h is Delaunay, then for each $B_h \subset M_h$:

$$\left| \sum_{E \subset B_h} (\theta_E - \pi) \| E \cap B_h \| - \int_{\Phi^{-1}(B_h)} H \, da \right| = O(\varepsilon)$$

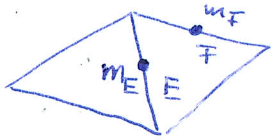
where $\Phi : S \to M_h$ is shortest distance map (cf. Sec. 3.3).

Remark: Similar convergence result for discrete weak Gauss curvature (defined as angle-defect).

Non-conforming FEM

Let $\{\phi_E\}_E$ be canonical basis fnc. set of Crouzeix-Raviart element on $\mathcal{M}_h$, i.e. $\phi_E(m_F) = \delta_{EF}$ and $\phi_E|_f$ affine $\forall f \in \mathcal{F}$.

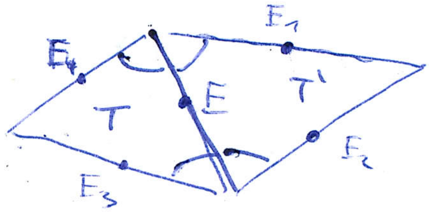
(Fncs in that space are continuous at edge midpoints!)



Lemma 2.4.5 in Ward '06:

$$\int_{\mathcal{M}_h} \Delta_h u \, \phi_E \, da = 2 \sum_{j=1}^{4} \cot(\measuredangle(E_j, E))\,(u_{E_j} - u_E),$$

where $u = \sum_E u_E \phi_E$, $u_E = u(m_E)$.



$$\int_{\omega_E} H_h N \, da := - \int_{\mathcal{M}_h} \Delta_h \mathrm{id} \, \phi_E \, da$$

$\omega_E = T \cup T'$.

$$= -2 \cos\frac{\Theta_E}{2} \, \|E\| \cdot N_E,$$

where N_E is ~~the~~ angle-bisecting edge normal at $E = T \cap T'$.

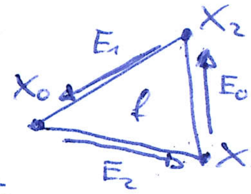
Triangle-averaged discrete shape operator

Sec. 3.1: discrete fFF $G = G_f \in \mathbb{R}^{2,2}$ constant on faces f.

Now: Aim at $H = H_f \in \mathbb{R}^{2,2}$ discrete sFF.

$\rightsquigarrow S_f \in \mathbb{R}^{2,2}$ by $S_f := G_f^{-1} H_f$ (cf. Sec. 2.1)

Let $f \in \mathcal{F}$ with local param. $X_f : \omega \to T(f)$.



$$\frac{\partial X_f}{\partial \xi_1} = X_1 - X_0 = E_2, \quad \frac{\partial X_f}{\partial \xi_2} = X_2 - X_0 = -E_1$$

Sec. 2.1: $h_{ij} = \langle \partial_i (n \circ x), \partial_j x \rangle_{\mathbb{R}^3}$, $i,j = 1,2$.

Hence: we get for the entries of $H = H_f$:

$$H_{11} = \langle dN(E_2), E_2 \rangle, \quad H_{12} = -\langle dN(E_2), E_1 \rangle$$

$$H_{21} = -\langle dN(E_1), E_2 \rangle \quad H_{22} = \langle dN(E_2), E_2 \rangle$$

N_E is angle-bisecting normal, i.e. $\alpha_r = \alpha_f = \frac{1}{2}$.

~~Hence ...~~

What is dN?

A continuous 1-form ω on a surface S is a mapping with $\omega(p) \in T_p^*S \;\; \forall p \in S$, where T_p^*S is dual of T_pS.

Cont. 1-forms are evaluated as integrals along (pw. diff.) curves $\gamma : [a,b] \to S$.

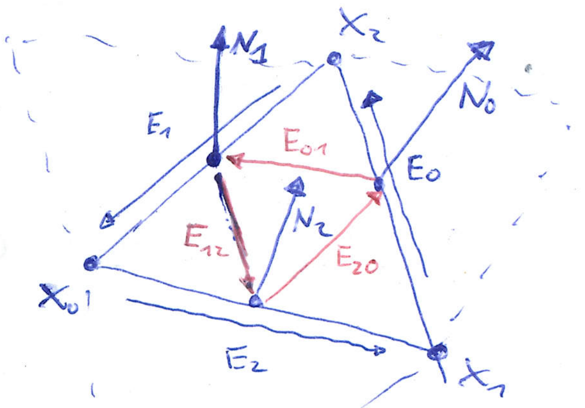
Ex: $\omega = df$, $f : S \to \mathbb{R}$ function

$$\Rightarrow \int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

$\rightarrow$ discrete 1-form are evaluated as integrals along discrete curves on M_h, i.e. along polygonal chains.

In detail: evaluate dN along the discrete curve E_{ij}!

Since $E_h = -2E_{ij}$ we get



$\boxed{N_i = N_{E_i}}$

$$dN(E_h) = -2\, dN(E_{ij})$$

$$= -2 \int_{E_{ij}} dN$$

$$= -2(N_j - N_i) = 2(N_i - N_j)$$

Hence we get for the entries H_{ij}:

$$H_{11} = 2\langle N_0, E_2\rangle + 2\langle N_1, E_0\rangle$$

$$H_{12} = H_{21} = 2\langle N_0, E_2\rangle$$

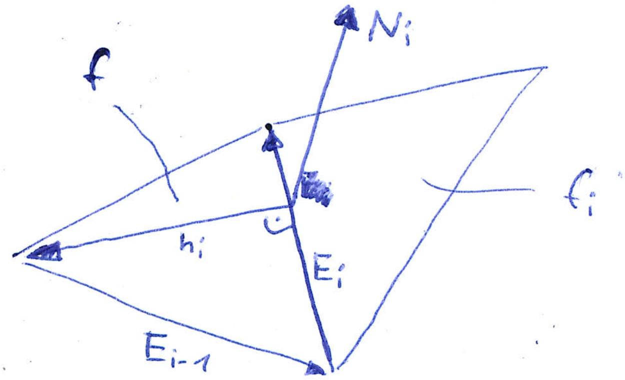
$$H_{22} = 2\langle N_0, E_2\rangle + 2\langle N_2, E_1\rangle$$

since $\langle N_i, E_i\rangle = 0$ and $E_0 + E_1 + E_2 = 0$.

$$\Rightarrow \quad H = H_f = 2\sum_{i=0}^{2} \langle N_i, E_{i-1}\rangle M_i$$

where (M_0, M_1, M_2) is basis of symm. 2×2-matrices given by $M_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

<u>Geometric interpretation</u>:



$$\theta_{E_i} = 2\cdot \sphericalangle(N_i, h_i)$$

$$h_i = -\beta E_i - E_{i-1}, \quad \beta \in [0,1].$$

$$\Rightarrow \quad \cos\frac{\theta_{E_i}}{2} = \left\langle N_i, \frac{h_i}{\|h_i\|}\right\rangle$$

$$= -\frac{1}{\|h_i\|}\langle N_i, E_{i-1}\rangle - \beta\|h_i\|^{-1}\underbrace{\langle N_i, E_i\rangle}_{=0}$$

Since $a_f = \frac{1}{2}\|h_i\|\cdot\|E_i\|$:

$$\langle N_i, E_{i-1}\rangle = -2\frac{a_f}{\|E_i\|}\cos\frac{\theta_{E_i}}{2}$$

$$\Rightarrow \quad H_f = -4 a_f \sum_{i=0}^{2} \frac{\cos\frac{\theta_i}{2}}{\|E_i\|} M_i$$

And finally: $\quad S_f := G_f^{-1} H_f \in \mathbb{R}^{2\times 2}$.

Discrete mean curvature on triangle

$\leadsto$ given as $\operatorname{tr} S_f$!

We have $\det G_f = 4 a_f^2 \leadsto \operatorname{tr}(G_f^{-1} M_i) = \dfrac{\|E_i\|}{4 a_f^2}$, $i=0,1,2$

$$\leadsto \operatorname{tr} S_f = \operatorname{tr}(G_f^{-1} H_f) = - \sum_{i=0}^{2} \frac{\cos \frac{\theta_i}{2}}{a_f} \|E_i\|$$

Remark: Non-conf. FEM : $\displaystyle\int_{\omega_E} H_h N \, da = -2 \cos \frac{\theta_E}{2} \|E\| \cdot N_E$

$$\leadsto \int_T H \, da := \frac{1}{2} \sum_{i=0}^{2} \left(-2 \cos \frac{\theta_E}{2} \|E\| \right)$$

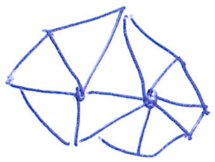
$$\leadsto H_f := \frac{1}{|T|} \int_T H \, da = - \sum_{i=0}^{2} \frac{\cos \frac{\theta_i}{2}}{a_f} \|E_i\|$$

4. Deformations of discrete shells

Aim: study deformation & deform. paths of discrete surfaces. Intuitive results require sound physical model!

Start in continuous setting:

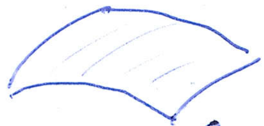
computationally | mathematically | physically



M_h

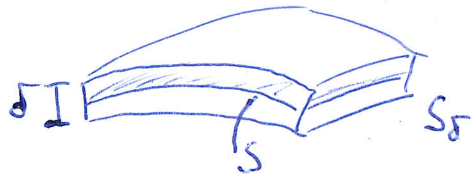
discrete surface / triangle mesh

$M_h \approx S$



S

smooth embedded surface $S \subset \mathbb{R}^3$



thin shell $S_\delta \subset \mathbb{R}^3$ with finite thickness $\delta > 0$ and midsurface S

Start with 3D elasticity for some solid body $\Omega_\delta \subset \mathbb{R}^3$.

Then consider $\delta \to 0$ based on either

(i) suitable notion of convergence (Γ-convergence)

(ii) apriori assumption on the deformation

4.1 Elasticity theory

References: several papers & books by Ciarlet;
D. Braess, "Finite elements".

Let $O \subset \mathbb{R}^3$ solid object, $\phi : O \to \mathbb{R}^3$ deformation.

Assume $\phi \in H^1(O, \mathbb{R}^3)$, $\det D\phi > 0$ a.e.

Postulate: there is an elastic deformation energy $W[\phi, O]$.

depends only on gradient of ϕ!

associated with ϕ.

Hyperelastic materials: $\exists$ energy density $W: \mathbb{R}^{3,3} \to \mathcal{O}$:

$$\mathcal{W}[\phi, \mathcal{O}] = \int_{\mathcal{O}} W(D\phi)\, dx$$

Frame indifference: $W(D\phi) = W(Q^T D\phi\, Q) \quad \forall Q \in SO(3)$

$\Rightarrow W(D\phi) = W(C[\phi])$, $C[\phi] = D\phi^T D\phi$ (Cauchy-Green strain tensor)

Elastic strain: $E[\phi] = \frac{1}{2}(C[\phi] - \mathbb{1})$

Assumption: $\mathcal{O}$ is isotropic material, i.e.

$$W(D\phi) = W(D\phi\, Q) \quad \forall Q \in SO(3)$$

Rivlin-Eriksen-Thm: Frame indiff & isotropy. Then:

W depends on singular values of $D\phi$

or, equivalently, on the invariants I_1, I_2, I_3 of $D\phi$

$$I_1 = \|D\phi\|_F, \quad I_2 = \|\operatorname{cof} D\phi\|_F, \quad I_3 = \det D\phi$$

where $\operatorname{cof} A = \det A \cdot A^{-T}$ for $A \in \mathbb{R}^{d,d}$; $\|A\|_F^2 = \operatorname{tr}(A^T A)$.

$\rightsquigarrow W(D\phi) = \hat{W}(I_1, I_2, I_3)$

Assume: isometries ϕ are local minimizers with zero energy!

$$\rightsquigarrow W(D\phi) = W(\underset{\overset{\shortparallel}{D\phi^T D\phi}}{\mathbb{1}}) = 0, \quad \partial_A W(\mathbb{1}) = 0$$

I_1, I_2, I_3 describe locally averaged change of length, area and volume, respectively.

Example:

$$W(D\phi) = \hat{W}(I_1, I_2, I_3) = a_1 I_1^p + a_2 I_1^q + \Gamma(I_3)$$

with $a_{1,2} \geq 0$, convex function $\Gamma : (0,\infty) \to \mathbb{R}$ with $\Gamma(I_3) \to \infty$ if $I_3 \to 0$ or $I_3 \to \infty$. (ensures local injectivity)

Mooney-Rivlin model: $p = q = 2$.

We shall make use of

$$W(D\phi) = \frac{\mu}{2} \|D\phi\|_F^2 + \frac{\lambda}{4}(\det D\phi)^2 - \left(\mu + \frac{\lambda}{2}\right) \log \det D\phi - \frac{d\mu}{2} - \frac{\lambda}{4}$$

for $d \in \{2,3\}$, s.t. $W(\mathbb{1}_d) = 0$.

Linear elasticity

$\phi(x) = x + u(x)$, u displacement, $\|u\|$ small!

Geometric linearization of elastic strain:

$$E[\phi] = \frac{1}{2}(D\phi^T D\phi - \mathbb{1}) = \frac{1}{2}\left((Du + \mathbb{1})^T (Du + \mathbb{1}) - \mathbb{1}\right)$$

$$= \underbrace{\frac{1}{2}(Du + Du^T)}_{=: \varepsilon[u]} + O(Du^T Du)$$

$=: \varepsilon[u]$ linearized elastic strain

A typical energy density in linearized, isotropic elasticity reads:

$$W^{lin}(D\phi) = W^{lin}(Du) = \frac{1}{2}\sigma : \varepsilon[u]$$

where $A : B = \operatorname{tr}(A^T B)$ and σ is the stress tensor given by

$$\sigma = \lambda \operatorname{tr}\varepsilon[u] \cdot \mathbb{1} + 2\mu\,\varepsilon[u] = \frac{E}{1+\nu}\left(\varepsilon[u] + \frac{\nu}{1-2\nu} \operatorname{tr}\varepsilon[u]\,\mathbb{1}\right)$$

which represents a linear material law

Physical parameters:
- λ, μ Lamé parameters (area, length)
- $E = \frac{\mu(3\lambda + 2\mu)}{\mu + \lambda}$ elastic modulus
- $\nu = \lambda / 2(\lambda + \mu)$ Poisson's ratio

Hence

$$W^{lin}(Du) = \frac{1}{2}\sigma : \varepsilon[u] = \frac{\lambda}{2}(\operatorname{tr}\varepsilon[u])^2 + \mu \operatorname{tr}(\varepsilon[u]^2)$$

The linear material law is often written in terms of the <u>elastic tensor</u> $\mathbb{C} : \mathbb{R}^{3,3} \to \mathbb{R}^{3,3}$, $\sigma = \mathbb{C}\,\varepsilon[u]$, i.e. $W^{lin}(Du) = \frac{1}{2}\mathbb{C}\varepsilon[u] : \varepsilon[u]$.

<u>Remark</u>: Nonlinear elastic density is invariant w.r.t. rigid body motions, i.e. $W^{lin}(D\phi) = 0$ if $\phi(x) = Qx + b$, $Q \in SO(3)$.
<u>But</u>: For W^{lin} only true in infinitesimal sense, i.e. W^{lin} is invariant w.r.t. linearized r.b.m., i.e. $W^{lin}(D\phi) = 0$ if $\phi(x) = Ax + b$, A skew-symmetric, $A = -A^T$.

<u>Variational setup</u>

$$\mathcal{E}[\phi] = \int_O W(D\phi)\,dx - \underbrace{\int_O F\circ\phi\,dx}_{\text{body force}} - \underbrace{\int_{\Gamma} G\circ\phi\,da}_{\text{boundary force}} \overset{!}{\to} \min, \quad \Gamma \subset \partial O$$

subject to b.c., or

$$\mathcal{E}^{lin}[u] = \frac{1}{2}\int_O \mathbb{C}\varepsilon[u] : \varepsilon[u]\,dx - \int_O f\cdot u\,dx - \int_\Gamma g\cdot u\,da \overset{!}{\to} \min$$

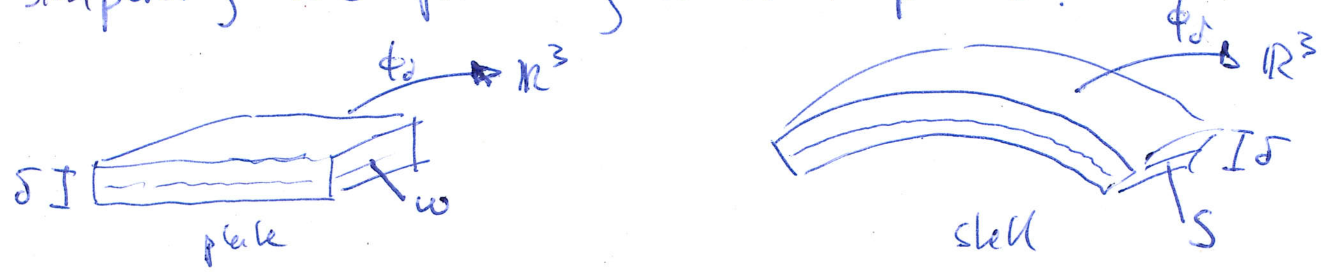
for functions $f : O \to \mathbb{R}^3$, $g : \Gamma \subset \partial O \to \mathbb{R}^3$ and suitable b.c.

However, we shall utilize W to define an elastic <u>dissimilarity measure</u> between shapes O_A and O_B:

$$d^2_{elast}(O_A, O_B) = \min_{\phi : \phi(O_A) = \phi(O_B)} \int_{O_A} W(D\phi)\,dx$$

Towards a 2D theory

For simplicity we primarily consider plates!



Let $\omega \subset \mathbb{R}^2$ with $\partial\omega$ Lipschitz, $\delta > 0$.

$$\Omega_\delta := \omega \times \left(-\tfrac{\delta}{2}, \tfrac{\delta}{2}\right), \qquad p_\delta = (\xi, z) \in \Omega_\delta .$$

Deformation $\phi_\delta \in H^1(\Omega_\delta; \mathbb{R}^3)$, elastic energy

$$\mathcal{W}[\phi_\delta, \Omega_\delta] = \int_{\Omega_\delta} W(D\phi_\delta)\, dp_\delta$$

Two categories of plate/shell theories (both starting with 3D elasticity!):

(i) ansatz-free rigorous convergence analysis (Γ-conv.)

(ii) restricting range of admissible deformations by further apriori assumptions.

<u>Next</u>: (ii) in Sec. 4.2, (i) in Sec. 4.3.

4.2 Linear elastic shells: Reissner–Mindlin and Kirchhoff–Love model

Consider plates here, i.e. $\omega \subset \mathbb{R}^2$, $\gamma := \partial\omega$.

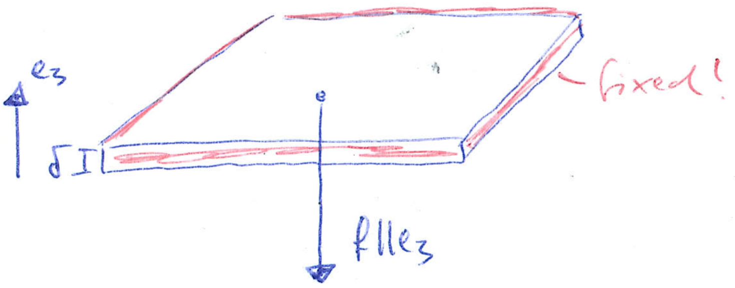
Body force $f : \Omega_\delta \to \mathbb{R}^3$, $f \parallel e_3$, $f(p_\delta) = f(\xi)$

No boundary forces.

Clamped b.c.: $\quad u(x) = u_\partial(x) \quad \forall x \in \Gamma_\delta = \gamma \times (-\frac{\delta}{2}, \frac{\delta}{2})$

and $u_\partial : \Gamma_\delta \to \mathbb{R}^3$ given.

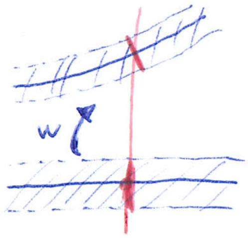
<u>Here:</u> $u_\partial \equiv 0$!



$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} : \Omega_\delta \to \mathbb{R}^3$$

$$\mathcal{E}^{lin}[u] = \frac{1}{2} \int_{\Omega_\delta} C\varepsilon[u] : \varepsilon[u] \, dx - \int_{\Omega_\delta} f \cdot u \, dx \longrightarrow \min!$$

s.t. $u = 0$ on $\Gamma_\delta := \partial\Omega_\delta$.



Apriori assumptions to derive 2D model:

(H1) normal segments undergo affine transformations

(H2) displacement u_3 only depends on ξ, i.e.
$u_3(p_\delta) = w(\xi)$ for some $w : \omega \to \mathbb{R}$

(H3) points on $\omega \times \{0\}$ are only deformed in e_3-direction,
i.e. $u_1(\xi, 0) = u_2(\xi, 0) = 0 \quad \forall \xi \in \omega$.

(H1) - (H3): $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(p_\delta) = -z\,\Theta(\xi)$

$$u_3(p_\delta) = w(\xi)$$

for $w: \omega \to \mathbb{R}$, $\Theta: \omega \to \mathbb{R}^2$.

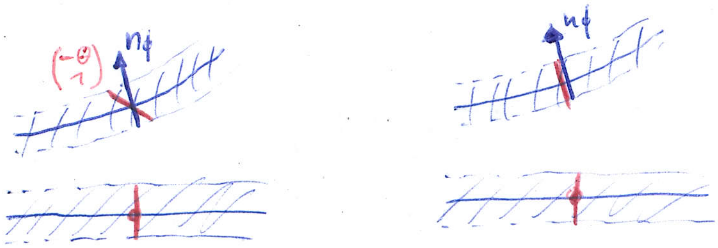
(H4) normal stress vanishes, i.e. $\sigma_{33} = 0$ $\quad (\sigma = \mathbb{C}\,\varepsilon[u])$

The hypotheses (H1)-(H4) lead to the <u>Reissner - Mindlin</u> - plate model.

Normal hypothesis / Kirchhoff-Love hypothesis:

(H5) deformed normal of ω is again normal to deformed mid surface.

(H1) - (H5) $\leadsto$ <u>Kirchhoff-Love</u> plate model



deformed normal $\hat{=} \begin{pmatrix} -\theta \\ 1 \end{pmatrix}$

normal of deformed midsurface n_ϕ

(R-M.) (K.-L.)

$\phi(x) = x + u(x)$, $\quad \partial_i \phi(\xi, 0) = \begin{pmatrix} e_i \\ \partial_i w(\xi) \end{pmatrix}$ deformed tangent vectors

$$\partial_3 \phi(\xi, 0) = \begin{pmatrix} -\theta(\xi) \\ 1 \end{pmatrix} \quad \text{deformed normal.}$$

Deformed tangents vectors are tangent to deformed midsurface!

(H5) $\Rightarrow$ $\partial_i \phi(\xi, 0) \perp \partial_3 \phi(\xi, 0)$ $\Rightarrow$ $\nabla w = \Theta$.

Reissner - Mindlin plate model

$$u(\xi,z) = \begin{pmatrix} -z\,\theta(\xi) \\ w(\xi) \end{pmatrix} \Rightarrow Du = \begin{pmatrix} -z D_\xi \theta & -\theta \\ (\nabla_\xi w)^T & 0 \end{pmatrix}$$

$$\Rightarrow \varepsilon[u] = \left(\begin{array}{c|c} -z\,\varepsilon[\theta] & \frac{1}{2}(\nabla_\xi w - \theta) \\ \hline \frac{1}{2}(\nabla_\xi w - \theta)^T & 0 \end{array}\right)$$

Hence $\varepsilon_{33}[u] = 0$. (H4) implies

$$0 = \sigma_{33} = \frac{E}{1+\nu}\left(\varepsilon_{33}[u] + \frac{\nu}{1-2\nu}\left(\varepsilon_{11}[u] + \varepsilon_{22}[u] + \varepsilon_{33}[u]\right)\right)$$

$$\Rightarrow \varepsilon_{11}[u] + \varepsilon_{22}[u] = 0 \Rightarrow \operatorname{tr}\varepsilon[u] = 0.$$

$$\Rightarrow \sigma : \varepsilon[u] = \frac{E}{1+\nu}\left(\varepsilon[u] : \varepsilon[u]\right).$$

$$\mathcal{E}^{lin}[u] = \mathcal{E}^{lin}[w,\theta] = \int_{\Omega_\delta} \frac{1}{2}\sigma : \varepsilon[u] - f u \, d\mu_\delta$$

$$= \frac{1}{2}\int_\omega \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{E}{1+\nu}\left(z^2\,\varepsilon[\theta]:\varepsilon[\theta] + |\nabla_\xi w - \theta|^2\right) - f(\xi)\,w(\xi)\,dz\,d\xi$$

$$= \frac{\delta^3 E}{24(1+\nu)}\int_\omega \varepsilon[\theta]:\varepsilon[\theta]\,d\xi + \frac{\delta E}{2(1+\nu)}\int_\omega |\nabla_\xi w - \theta|^2\,d\xi$$

$$- \delta\int_\omega f w \, d\xi$$

$$= \frac{\delta^3}{2}\,a(\theta,\theta) + \gamma\delta\,\|\nabla_\xi w - \theta\|_{L^2}^2 - \delta\,(f,w)_{L^2}$$

with $\gamma = \frac{E}{2(1+\nu)}$ and $a(\theta,\psi) := \frac{E}{12(1+\nu)}\int_\omega \varepsilon[\theta]:\varepsilon[\psi]\,d\xi$

Euler-Lagrange for corresponding optim. pb.:

Find $(\theta, w) \in (H^1_0(\omega))^3$ s.t.

$$\delta^3 a(\theta,\psi) + 2\delta\gamma\,(\theta - \nabla_\xi w, \psi)_{L^2} = 0 \quad \forall \psi \in (H^1_0(\omega))^2$$

$$2\delta\gamma\,(\nabla_\xi w, \nabla_\xi \vartheta)_{L^2} = (f,\vartheta)_{L^2} \quad \forall \vartheta \in H^1_0(\omega)$$

where we have used $\operatorname{div}\theta = \operatorname{tr}\varepsilon[\theta] = 0$ since $\operatorname{tr}\varepsilon[u] = 0$.

As $\vartheta \in H^1_0(\omega)$ we have $(\theta, \nabla_\xi \vartheta)_{L^2} = -(\operatorname{div}\theta, \vartheta)_{L^2} = 0$.

<u>Remark</u>: Under Kirchhoff-Love (H5):

$$\mathcal{E}^{lin}[u] = \mathcal{E}^{lin}[w] = \frac{\delta^3 E}{24(1+\nu)} \int_\omega |D^2 w|^2 \, d\xi - \delta \int_\omega f w \, d\xi$$

~~Saddle point problem & mixed methods~~

$\boxed{\text{Kirchhoff-Love plate model}}$

Standard FEM approach requires H^2-conforming elements!

$\rightarrow$ saddle point formulation $\quad (\delta = 1$ here$)$

Minimize $\frac{1}{2} a(\theta,\theta) - (f,w)_{L^2}$ s.t. $\theta = \nabla_\xi w$

$\rightarrow$ Lagrange functional $\mathcal{L} : X \times M \to \mathbb{R}$:

$$\mathcal{L}(w,\theta;\lambda) = \frac{1}{2} a(\theta,\theta) - (f,w)_{L^2} + (\lambda, \nabla_\xi w - \theta)_{L^2}$$

with $(w,\theta) \in X$ and "Lagrange multiplier" $\lambda \in M$.

Necessary conditions for saddle point:

$$\forall (v,\psi) \in X: \quad 0 = \partial_{(w,\theta)} \mathcal{L}[\cdot](v,\psi) = a(\theta,\psi) + (\nabla_\xi v - \psi, \lambda)_{L^2} - (f,v)_{L^2}$$

$$\forall \mu \in M: \quad 0 = \partial_\lambda \mathcal{L}[\cdot](\mu) = (\nabla_\xi w - \theta, \mu)_{L^2}$$

To solve these equations one can choose different spaces for X and $\mathcal{M}$, respectively.

1) $\mathcal{X} = H^2_0(\omega) \times (H^1_0(\omega))^2$ and $\mathcal{M} = H^{-1}_0(\omega)$.

2) $\mathcal{X} = (H^1_0(\omega))^3$ and $\mathcal{M} = H(\mathrm{div}, \omega) = \{ \gamma \in H^{-1}(\omega) : \mathrm{div}\, \gamma \in H^{-1}(\omega) \}$

One can show existence in both cases!

4.3 Membrane and bending energies by Γ-convergence

Now: ansatz-free derivation!

Remark: Will focus on qualitative results of convergence analysis!

Consider plates, i.e. $\omega \subset \mathbb{R}^2$ with $\partial\omega$ Lipschitz, $\Omega_\delta = \omega \times (-\frac{\delta}{2}, \frac{\delta}{2})$, $\phi_\delta \in H^1(\Omega_\delta, \mathbb{R}^3)$, and

$$\mathcal{W}[\phi_\delta, \Omega_\delta] = \int_{\Omega_\delta} W(D\phi_\delta)\, dp_\delta$$

for W frame-indifferent which is minimized on $SO(3)$ and $W(\mathbb{1}) = 0$ and $W(F) = \infty$ if $\det F \leq 0$.

Additionally: some regularity and growth conditions.

Ex:
- $W(F) = \mathrm{dist}^2(F, SO(3)) \approx \frac{1}{4} \| F^T F - \mathbb{1} \|_F^2$
- St Venant-Kirchhoff material (isotropic!)

  $$W(F) = \frac{\lambda}{8} \left(\mathrm{tr}(F^T F - \mathbb{1}) \right)^2 + \frac{\mu}{4}\, \mathrm{tr}\left(F^T F - \mathbb{1}\right)^2$$

  $\leadsto$ deduced from W^{lin} by replacing ε by strain $E[\phi_\delta]$.
- $W(F) = a \|F\|_F^2 + b \|\mathrm{cof}\, F\|_F^2 + c(\det F)^2 - d \log \det F + e$

An excursion on Γ-convergence

Def. 4.1 (Γ-convergence)

Let (X,d) metric space, $F_j : X \to \mathbb{R}$ a sequence of functionals and $F : X \to \mathbb{R}$. Then $F_j \xrightarrow{\Gamma} F$ w.r.t. d, if

(i) liminf-cond.: $\forall (x_j)_j \subset X$, $d(x_j, x) \to 0$ for $x \in X$:

$$F[x] \leq \liminf_{j\to\infty} F_j(x_j]$$

(ii) limsup-cond: $\forall x \in X \ \exists (x_j)_j \in X$ with $d(x_j, x) \to 0$ and

$$F(x] \geq \limsup_{j\to\infty} F_j(x_j]$$

(recovery sequence)

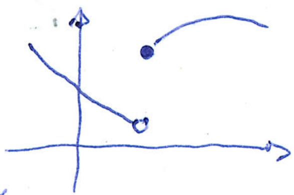
Remarks:

(i) If $F_j \xrightarrow{\Gamma} F$, then F is lsc. wrt. d,

i.e. if $d(x_j, x) \to 0$ then $F(x] \leq \liminf_{j\to\infty} F[x_j]$.

(ii) If $F_j \xrightarrow{\Gamma} F$, G continuous, then

$F_j + G \xrightarrow{\Gamma} F_j + G$



not lsc.!

(iii) If $F_j \xrightarrow{\Gamma} F$, $G_j \xrightarrow{\Gamma} G$ then not necessarily

$F_j + G_j \xrightarrow{\Gamma} F + G$!

(iv) If $F_j \equiv F$ then not necessarily $F_j \xrightarrow{\Gamma} F$ (limit is lsc.!)

Remark on lsc.: (Direct methods of Calculus of Variations)

If X is reflexive Banach space, F is weakly lsc., and coercive, then F attains its minimum in X.

Example: $X = \mathbb{R}$, $d(x,y) = \|x-y\|$.

$F_j(x) := \sin(jx)$, $F(x) := -1$. $\Rightarrow F_j \xrightarrow{\Gamma} F$.

(i) ✓

(ii) Let $x \in \mathbb{R}$. Define $x_j \in \mathbb{R}$ s.t. $d(x,x_j) = \min\{d(x,y) : \sin(jy) = -1\}$

$\Rightarrow d(x_j,x) \to 0$ and $-1 \geq F_j[x_j]$.

A sequence $(F_j)_j$ is said to be equi-mildly coercive if there is some compact set $K \subset X$ with

$$\inf_{x_j \in X} F_j(x_j] = \inf_{x_j \in K} F_j(x_j] \quad \forall j$$

Properties of Γ-convergence related to existence of minimizers:

(i) If $F_j \xrightarrow{\Gamma} F$, x_j minimizer of F_j, $x_j \to x$.

Then x is minimizer of F.

(ii) If $F_j \xrightarrow{\Gamma} F$, $(F_j)_j$ equi-mildly coercive then there is a minimizer x of F with $F(x] = \lim_{j\to\infty} \inf_{x_j \in X} F_j[x_j]$.

Moreover, if $(x_j)_j$ is minimizing sequence, each accumulation point is minimum point of F.

Proof (i): Assume $\exists x' \in X$ with $F(x'] < F(x]$. Let $x'_j \to x'$.

$$F(x'] \underset{4.1(ii)}{\geq} \limsup F_j(x'_j] \geq \liminf F_j(x'_j] \underset{x_j \text{ min. of } F_j}{\geq} \liminf F_j[x_j]$$

$$\underset{4.1(i)}{\geq} F[x]. \ \lightning$$

$\leadsto$ (i) requires existence and convergence of minimizers x_j,

(ii) provides existence of minimizer x without these conditions.

Application of Γ-convergence to dimension reduction

Parameter $\delta \to 0$, $\phi_\delta \in H^1(\Omega_\delta, \mathbb{R}^3)$ live on different spaces Ω_δ!

$\to$ rescaling: $p_\delta = (\xi, z) \in \Omega_\delta \mapsto p = (\xi, \delta z) \in \Omega_1$

and gradient transformation $D_\delta = (D_\xi, \frac{1}{\delta}\partial_z)$

$$\to \quad \mathcal{W}^*[\phi_\delta, \Omega_1] = \delta \int_{\Omega_1} W(D_\delta \phi_\delta(p))\, dp$$

Observation: different scaling in δ of $\mathcal{W}^*$ for different loads:

$\sim \delta$ $\qquad\qquad$ $\sim \delta^3$

Γ-limit for membrane model

Assume $\qquad c_1 \|F\|^2 - c_2 \le W(F) \le c_3 \|F\|^2 + c_4$

and define $\qquad \mathcal{W}^\delta_{mem}[\phi_\delta, \Omega_\delta] := \frac{1}{\delta} \mathcal{W}^*[\phi_\delta, \Omega_1]$

Thm 4.2 (Membrane Γ-Limit, [LeDret & Raoult '95, '96])

$$\mathcal{W}^\delta_{mem}[\phi_\delta, \Omega_\delta] \xrightarrow[\text{weak } H^1]{\Gamma} \mathcal{W}^0_{mem}[\phi, \omega] = \int_\omega QW_{2D}(D\phi)\, d\xi$$

where $\phi \in H^1(\omega, \mathbb{R}^3)$ is deformation of midplane $\omega \subset \mathbb{R}^2$.

The density $QW_{2D}: \mathbb{R}^{3,2} \to \mathbb{R}$ arises from double relaxation process:

(1) For $F \in \mathbb{R}^{3,2}$: $W_{2D}(F) = \min_{b \in \mathbb{R}^3} W([F|b])$

(2) Quasi-convex envelope: $QW_{2D}(F) = \inf_{\ell} \left\{ ⨍_\omega W_{2D}(F + D\ell)\, dx \right\}$

Remarks (1) If W fulfills growth condition, then relaxation of $\int_\Omega W(D\phi)\,dx$ is given by $\int_\Omega QW(D\phi)\,dx$ (wrt. weak H^1-topol.)
Relaxation of F is defined as $\operatorname{rel} F := \sup_G \{ G \text{ lsc} : G \leq F \}$.

(2) A corresponding result holds for a curved reference domain, i.e. $\omega \subset \mathbb{R}^2$ is replaced by $S \subset \mathbb{R}^3$.

(3) For general W it is very complex to compute QW_{2D}.

Qualitative properties of Γ-limit

W frame-indiff & isotropic.

$$\mathcal{W}^0_{mem}[\phi, S] \cong \int_S W(\underbrace{D\phi^T D\phi}_{=: C[\phi]})\,da$$

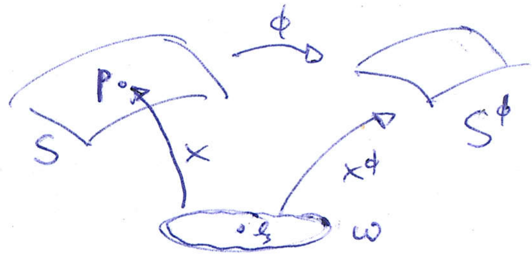
$=: C[\phi]$ measures distortion of tangent vectors

$$g_p(C[\phi]V, W) = g_{\phi(p)}(D\phi V, D\phi W), \quad V, W \in T_pS.$$

There is a 2×2-matrix representation $G[\phi]$ of $C[\phi]$!

$\phi = x^\phi \circ x^{-1}$ (locally)

$\rightsquigarrow C[\phi] \triangleq \boxed{g^{-1} g_\phi =: G[\phi]}$



$\phi = x^\phi \circ x^{-1}$

Define:

$$\boxed{\mathcal{W}_{mem}[S, \phi] := \int_S W_{mem}(G[\phi])\,da}$$

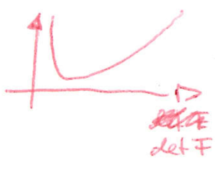
with

local change of length

$$W_{mem}(F) = \frac{\mu}{2}\,\mathrm{tr}(F) + \frac{\lambda}{4}\det F - \left(\frac{\mu}{2} + \frac{\lambda}{4}\right)\log \det F + c$$

local change of area

with $c \in \mathbb{R}$ s.t. $W_{mem}(\mathbb{1}) = 0$.



Γ-limit for bending model (plates)

$$\mathcal{W}^\delta_{\text{bend}}[\phi_\delta, \Omega_\delta] := \frac{1}{\delta^3} \int_{\Omega_\delta} W(D\phi_\delta)\, dp_\delta = \frac{1}{\delta^2} \int_{\Omega_1} W(D_\delta \phi_\delta)\, dp$$

with $W : \mathbb{R}^{3,3} \to \mathbb{R}$ continuous & frame-indifferent, growth condition $W(F) \geq c \operatorname{dist}^2(F, SO(3))$ and $W(F) = 0$ for $F \in SO(3)$.

For simplicity: assume that W is isotropic with

$$W_{,FF}(\mathbb{1})(G,G) = \lambda (\operatorname{tr} G)^2 + \frac{\mu}{2} \operatorname{tr}\left((G+G^T)^2\right)$$

which is fulfilled by the St-Venant-Kirchhoff density.

<u>Thm 4.3</u> (Bending Γ-limit for plates, [Friesecke, James, Müller, '02])

$$\mathcal{W}^\delta_{\text{bend}}[\phi_\delta, \Omega_\delta] \xrightarrow[H^1]{\Gamma} \mathcal{W}^0_{\text{plate}}[\phi, \omega]$$

with $$\mathcal{W}^0_{\text{plate}}[\phi, \omega] = \begin{cases} \frac{1}{24} \int_\omega 2\mu \operatorname{tr}(h^2) + \frac{\lambda\mu}{\mu + \lambda/2} (\operatorname{tr} h)^2 \, d\xi, & (*) \\ +\infty & \text{, else} \end{cases}$$

(*) for isometries $\phi : \omega \to \mathbb{R}^3$.

<u>Remark</u>: $h = h[\phi] = Dn^T D\phi$ where $\phi : \omega \to \mathbb{R}^3$ is parametrization of deformed plate $\phi(\omega) \subset \mathbb{R}^3$, $n \parallel \phi_{,1} \times \phi_{,2}$.

$\phi : \omega \to \mathbb{R}^3$ is isometry iff. $(g_\phi)_{ij} = \phi_{,i} \cdot \phi_{,j} = \delta_{ij}$

~~Idea~~ Sketch of proof for recovery sequence:

For simplicity: $W = W^{StVK}$ with $\lambda = 0$, $\mu = 1$

$\rightsquigarrow W(F) = \frac{1}{4} \| F^T F - \mathbb{1} \|^2$

Consider $\mathcal{W}^{\delta}_{\text{shell}}[\phi_\delta, \Omega_\delta] = \delta \int_{\Omega_1} W(D_\delta \phi_\delta)\, dp$, $D_\delta = (D_\xi, \frac{1}{\delta}\partial_z)$.

Let $\phi : \omega \to \mathbb{R}^3$ deformation of ω.

Define sequence $\phi_\delta : \Omega_1 \to \mathbb{R}^3$ by $\phi_\delta(\xi, z) = \phi(\xi) + \delta z n(\xi)$.

where $\langle \partial_{\xi_h} \phi(\xi), n(\xi) \rangle = 0$, $h = 1,2$.

Then

$$D_\delta \phi_\delta = [D_\xi \phi, n] + \delta z [D_\xi n, 0] \in \mathbb{R}^{3,3}$$

Second FF is given by $h = h[\phi] = (D_\xi \phi)^T D_\xi n$ (symmetric!),

first FF $g = g[\phi] = (D_\xi \phi)^T D_\xi \phi$, $(D_\xi n)^T n = 0$, $|n|^2 = 1$.

$$\Rightarrow (D_\delta \phi_\delta)^T (D_\delta \phi_\delta) = \left[\begin{array}{c|c} g + 2\delta z h + \delta^2 z^2 r & 0 \\ \hline 0 & 1 \end{array}\right]$$

with $r := (D_\xi n)^T D_\xi n$.

$$\Rightarrow \mathcal{W}^{\delta}_{\text{shell}}[\phi_\delta, \Omega_\delta] = \frac{1}{\delta^3} \cdot \delta \cdot \int_{\Omega_1} \frac{1}{4} \| (D_\delta \phi_\delta)^T D_\delta \phi_\delta - \mathbb{1} \|^2 \, dp$$

[integration in z] $\rightarrow$:

$$= \int_\omega \frac{1}{4\delta^2} \| g - \mathbb{1} \|^2 + \frac{1}{24} (g - \mathbb{1}) : r + \frac{1}{12} \| h \|^2 + \frac{\delta^2}{20} \| r \|^2 \, d\xi$$

$$\xrightarrow{\delta \to 0} \begin{cases} \frac{1}{12} \int_\omega \| h \|^2 \, d\xi, & \text{if } \| g - \mathbb{1} \|^2 = 0 \\ +\infty & \text{, else} \end{cases}$$

which corresponds to $\mathcal{W}^{0}_{\text{plate}}[\phi, \omega]$ for $\lambda = 0$ and $\mu = 1$.

Γ-limit for ~~[illegible]~~ the bending model (shells)

$\omega \subset \mathbb{R}^2$ replaced by $S \subset \mathbb{R}^3$, $\Omega_\delta = \omega \times (-\frac{\delta}{2}, \frac{\delta}{2})$ by

$$S_\delta = \{ p + z\, n(p) \mid p \in S,\ z \in (-\tfrac{\delta}{2}, \tfrac{\delta}{2}) \}.$$

Thm 4.4 (Bending Γ-limit for shells, [Friesecke et al. '03])

$$\frac{1}{\delta^3} \int_{S_\delta} W(D\phi_\delta)\, dp_\delta \xrightarrow[H^1]{\Gamma} \mathcal{W}^0_{shell}[\phi, S],$$

with

$$\mathcal{W}^0_{shell}[\phi, S] = \begin{cases} \frac{1}{24} \int_S \min_{b \in \mathbb{R}^3} Q\left(S^{rel}_\phi(p) + b \otimes n(p)\right) da, & \phi \in \mathcal{A} \\ +\infty, & \text{else} \end{cases}$$

with quadratic form $Q(G) = W_{,FF}(\mathbb{1})(G, G)$ and admissible set $\mathcal{A} = \{ \phi \in H^2(S, \mathbb{R}^3) : (D_{tan}\phi)^T (D_{tan}\phi) = \mathbb{1} \text{ a.e.} \}$

where $D_{tan}\phi \in \mathbb{R}^{3,2}$ can be extended to $Q \in SO(3)$.
Relative shape op: $S^{rel}_\phi(p) : T_pS \to T_pS$.

Define:

$$\boxed{\mathcal{W}_{bend}[\phi, S] := \int_S \alpha (\operatorname{tr} S^{rel}_\phi)^2 + \beta \| S^{rel}_\phi \|_F^2\, da}$$

with coefficients $\alpha, \beta \in \mathbb{R}$. Usually $\alpha \in \{0, 1\}$ and $\beta = 1 - \alpha$.
Using the 2×2-matrix representation s^{rel}_ϕ of S^{rel}_ϕ, we get

$$\int_S (\operatorname{tr} S^{rel}_\phi)^2\, da = \int_\omega (\operatorname{tr} s^{rel}_\phi)^2 \sqrt{\det g}\, d\xi$$

$$\int_S \| S^{rel}_\phi \|_F^2\, da = \int_\omega \operatorname{tr}\left((s^{rel}_\phi)^2\right) \sqrt{\det g}\, d\xi$$

Full elastic model & dissimilarity measure

Given a surface $S \subset \mathbb{R}^3$ representing a physical shell with thickness $\delta > 0$. We define: for $\phi : S \to \mathbb{R}^3$:

$$\boxed{\mathcal{W}_S[\phi] = \int_S W_{mem}(G(\phi)) + \eta\, W_{bend}(S_\phi^{rel})\, da}$$

where $\eta \sim \delta^2$, $W_{bend}(\mathcal{F}) = \alpha (\operatorname{tr} \mathcal{F})^2 + \beta \|\mathcal{F}\|_F^2$.

<u>Note</u>: $\mathcal{W}_S[\phi] = 0$, $\partial \mathcal{W}_S[\phi] = 0$ <u>iff</u> ϕ is r.b.m.

Dissimilarity measure

$$d^2_{shell}(S_A, S_B) := \min_{\phi : \phi(S_A) = S_B} \mathcal{W}_{S_A}[\phi].$$

<u>Remark</u>: In general this might not be well-defined! However, in discrete shell space we can ensure well-definedness due to 1-to-1-correspondence of meshes.

<u>Remark</u>: Note the relation of d^2_{shell} to the Fund. Thm. of Surfaces: $d^2_{shell}(S_A, S_B) = 0$ <u>iff</u> S_A and S_B are congruent.

4.4 Discrete shells and discrete deformation energies

$\mathcal{M}_h$ discrete surface $\;=:\;$ discrete shell

topology / connectivity $\oplus$ geometry

$(\mathcal{V}, \mathcal{F})$ $\qquad\qquad$ $E: \mathcal{V} \to \mathbb{R}^3$

Def 4.6 (Dense correspondence / discret deformation)

Two discrete shells are in dense correspondence if they share the same topology / connectivity.
Given two shells S and $\tilde{S}$ with embeddings $E, \tilde{E}: \mathcal{V} \to \mathbb{R}^3$, a discrete deformation $\Phi: S \to \tilde{S}$ is the unique pw affine mapping defined by $\Phi(E(v_i)) := \tilde{E}(v_i) \quad \forall i = 1, \dots, |\mathcal{V}|$.

Remark: If $X = (1-\xi_1-\xi_2)X_i + \xi_1 X_j + \xi_2 X_k \in T(f)$ with $T(f) = \{X_i, X_j, X_k\} \subset S$, then

$$\Phi(X) = (1-\xi_1-\xi_2)\tilde{X}_i + \xi_1 \tilde{X}_j + \xi_2 \tilde{X}_k$$

with $\tilde{T}(f) = \{\tilde{X}_i, \tilde{X}_j, \tilde{X}_k) \subset \tilde{S}$.

Discrete membrane model

Let $S, \tilde{S}$ be in dense correspondence, $\Phi: S \to \tilde{S}$ discrete deform.
Discrete distortion tensor (elementwise constant!)

$$\mathcal{G}[\Phi]\big|_f := (G_f)^{-1}\tilde{G}_f \in \mathbb{R}^{2,2}$$

Discrete membrane energy, $\tilde{S} = \Phi(S)$:

$$\mathcal{W}_{mem}[S, \tilde{S}] = \int_S W_{mem}(\mathcal{G}[\Phi])\,da = \sum_{f \in \mathcal{F}} a_f\, W_{mem}(\mathcal{G}[\Phi]\big|_f)$$

<u>Remark</u>: One point quadrature rule sufficient!

$$W_{mem}(A) = \frac{\mu}{2} \operatorname{tr} A + \frac{\lambda}{4} \det A - \left(\frac{\mu}{2} + \frac{\lambda}{4}\right) \log \det A - \mu - \frac{\lambda}{4}.$$

$\operatorname{tr} G[\Phi]$ controls local change of length

$\det G[\Phi]$ ——— " ——— area.

Note that $W_{mem}(\mathbb{1}) = 0$, $\partial_A W_{mem}(\mathbb{1}) = 0$.

<u>Discrete bending model</u>

shape operators in 2x2-matrix repres.

$$\mathcal{W}_{bend}[S, \tilde{S}] = \int_S W_{bend}(S - S_\Phi)\, da$$

$$= \sum_{f \in \mathcal{F}} a_f \cdot \left[\alpha \operatorname{tr}\left((S_f - \tilde{S}_f)^2\right) + \beta \left(\operatorname{tr}(S_f - \tilde{S}_f)\right)^2 \right]$$

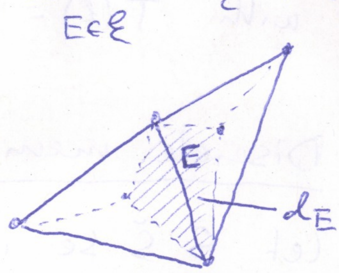
with $\alpha, \beta \in \mathbb{R}$, usually $\alpha \in \{0, 1\}$, $\beta = 1 - \alpha$.

<u>Simplified discrete bending model</u>

Let $\Phi : S \to \tilde{S}$ be an <u>isometric</u> deformation!

$$\int_S \operatorname{tr} S\, da = \sum_{f \in \mathcal{F}} a_f \cdot \operatorname{tr} S_f = \sum_{f \in \mathcal{F}} \sum_{i=0}^{2} -\cos\frac{\theta_i}{2} |E_i| = -2 \sum_{E \in \mathcal{E}} \cos\frac{\theta_E}{2} |E|.$$

$$= \sum_{E \in \mathcal{E}} d_E \left(\frac{-2\cos\frac{\theta_E}{2}}{d_E} |E| \right)$$



$$\leadsto \int_S \operatorname{tr}(S - \tilde{S})\, da \approx \sum_{E \in \mathcal{E}} d_E \left(\frac{\theta_E - \theta_{\tilde{E}}}{d_E} |E| \right)$$

1) isometry: $d_E = \tilde{d}_E$, $|E| = |\tilde{E}|$

2) Taylor (about $\theta = \pi$): $f(\theta) = -2\cos\frac{\theta}{2} \approx (\theta_E - \pi) + O(|\pi - \theta_E|^3)$

$$\overset{\text{spatial averaging}}{\leadsto} \mathcal{W}^{DS}_{bend}[S, \tilde{S}] := \sum_{E \in \mathcal{E}} \frac{(\theta_E - \theta_{\tilde{E}})^2}{d_E} |E|^2$$

("Discrete Shells" bending energy)

Discrete dissimilarity measure

Def. 4.7 (Discrete diss. measure)

Given two discrete shells $S, \tilde{S}$ in dense correspondence.

$$\mathcal{W}[S, \tilde{S}] = \mathcal{W}_{mem}[S, \tilde{S}] + \eta\, \mathcal{W}_{bend}[S, \tilde{S}],$$

with $\eta \sim \delta^2$ bending weight.

5 The shape space of discrete shells

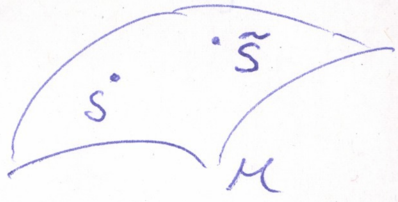
So far: single shells / triangular meshes

Now: consider space of <u>all</u> discrete shells,

i.e. all discrete shells that are in dense correspondence!

No mathematical structure apriori, no linear vector space!

General setup: Riemannian manifold $\mathcal{M}$!



~~Since we fix the topology / connectivity $\mathcal{M} = \mathbb{R}^{3n}$, $n = |\mathcal{N}|$.~~

Riemannian structure is induced by either

(i) a Riemannian metric $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ $\forall p \in \mathcal{M}$.

(ii) a (squared) Riemannian distance measure

$$\mathrm{dist}^2 : \mathcal{M} \times \mathcal{M} \to \mathbb{R}.$$

In physical shape spaces it is hard to define (i) apriori.

<u>But</u>: physical dissimilarity measure is <u>local</u> approximation of dist^2! $\rightsquigarrow$ metric g is implicitly induced.

Why a Riemannian manifold?

$\rightsquigarrow$ Once a Riemannian structure is derived (by (i) or (ii)), lots of usefull geometric operations are readily available:

(1) geodesics (locally shortest path) $\rightsquigarrow$ morphing / interpolation

(2) exponential map $\rightsquigarrow$ extrapolation

(3) logarithm $\rightsquigarrow$ linear representation of nonlinear variations

(4) parallel transport $\rightsquigarrow$ detail transfer along path

However: Need a time-discretization!

$\rightsquigarrow$ will make use of variational time-discretization introduced by Rumpf & Wirth:

(Time-discrete) geodesics are minimizers of (time-discrete) path energy!

Then discretization of (2)-(4) follows immediately.

5.1 Geodesic calculus on a Riemannian manifold

Usually: geodesics are defined as solution of the geodesic equation, i.e. $\frac{D}{dt}\dot{y}(t) = 0$ for $y : I \to \mathcal{M}$, where $\frac{D}{dt}$ is covariant derivative along y.

However: we define geodesics as minimizers of path energy!

Setup: $\mathcal{M}$ finite-dimensional manifold, $x : \omega \subset \mathbb{R}^d \to \mathcal{M}$.

Riemannian metric $g : p \mapsto g_p$ is mapping s.t. $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is bilinear, symmetric, positive-definite form which varies smoothly in p.

Set $\underset{g(p) = g(\xi)}{g} \in \mathbb{R}^{d \times d}$ with $g_{ij} := g_{\underset{x(\xi)}{p}}(X_i, X_j)$, $X_i := \partial_{\xi_i} x(\xi)$.

Path energy and geodesics: Let $y : [0,1] \to \mathcal{M}$ smooth path.

$$\mathcal{L}[y] = \int_0^1 \sqrt{g_{y(t)}(\dot{y}(t), \dot{y}(t))}\, dt \qquad \text{(length)}$$

$$\mathcal{E}[y] = \int_0^1 g_{y(t)}(\dot{y}(t), \dot{y}(t))\, dt \qquad \text{(path energy)}$$

Remark: $\mathcal{L}$ is invariant wrt. to reparametrization of y, $\mathcal{E}$ is not!

Cauchy-Schwarz:

$$L[y] \leq \sqrt{\mathcal{E}[y]} \quad \text{with "=" iff. } g_{y(t)}(\dot{y},\dot{y}) = \text{const}$$

Will see: geodesics have this <u>constant speed property</u>!

<u>Def. 5.1</u> (Geodesic path)

For $y_A, y_B \in \mathcal{M}$ a minimizer of $\mathcal{E}$ among all paths $y:[0,1] \to \mathcal{M}$ with $y(0) = y_A$, $y(1) = y_B$, is denoted as <u>geodesic path</u> connecting y_A and y_B.

<u>Remark:</u> y minimizer of $\mathcal{E} \Rightarrow y$ minimizer of L.

Other implication is wrong!

Rumpf & Wirth have shown that Def 5.1 is well-defined for general infinite dim. manifolds under suitable assumptions on $\mathcal{M}$ and g. In particular, existence and (local) uniqueness!

$$\operatorname{dist}(y_A, y_B) := \min_{\substack{y:[0,1]\to\mathcal{M} \\ y(0)=y_A,\, y(1)=y_B}} L[y] = \sqrt{\min_{-\,\shortparallel\,-} \mathcal{E}[y]}$$

(Riemannian distance)

Covariant derivative

$\leadsto$ differentiation of vector fields on $\mathcal{M}$.

If $\dim \mathcal{M} = d < \infty$: work with coordinates in $\Omega \subset \mathbb{R}^d$.

For general manifolds: seek for coordinate-free representation!

Let $x : \Omega \subset \mathbb{R}^d \to M$ parametrization, $\xi \in \Omega$, $p = x(\xi)$.

Then $x_{,k} = \partial_{\xi_k} x \in T_pM$. But

$$x_{,ij} = \partial_{\xi_j} \partial_{\xi_i} x = \sum_{k=1}^{d} \Gamma_{ij}^{k} x_k + \sum_l \beta_l n_l$$

Christoffel symbols

where $(n_l)_l$ is basis of $(T_pM)^\perp$.

$$\Rightarrow x_{,ij} \cdot x_k = \sum_{l=1}^{d} \Gamma_{ij}^{l} g_{lk} \qquad (*)$$

Symmetry of $x_{,ij}$ induces symmetry of Γ_{ij}^k in (i,j).

<u>Prop. 5.4</u>. Let $g^{-1} = (g^{ij})_{ij}$ inverse of g. Then

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{l=1}^{d} g^{lk} \left(g_{ij,l} - g_{li,j} + g_{jl,i} \right) \qquad (**)$$

<u>Proof</u>: Differentiation of $g_{ij} = x_{,i} \cdot x_{,j}$ with $(*)$:

① $g_{ij,k} = x_{,ik} \cdot x_{,j} + x_{,i} \cdot x_{,jk} = \sum_l \Gamma_{ik}^{l} g_{lj} + \sum_l \Gamma_{jk}^{l} g_{li}$

② $g_{ki,j} =$ ——— " ——— $= \sum_l \Gamma_{kj}^{l} g_{li} + \sum_l \Gamma_{ij}^{l} g_{lk}$

③ $g_{jk,i} =$ ——— " ——— $= \sum_l \Gamma_{ji}^{l} g_{lk} + \sum_l \Gamma_{ki}^{l} g_{lj}$

$\to$ Summing $(① - ② - ③)$:

$$g_{ij,k} - g_{ki,j} + g_{jk,i} = 2 \sum_{l=1}^{d} \Gamma_{ki}^{l} g_{lj}$$

$\to$ multiplication with g^{-1}, i.e. $\sum_k (-) g^{lk}$ finishes proof. □

Towards a coordinate-free representation:

$$\Gamma_p : T_pM \times T_pM \to T_pM, \quad \Gamma_p(X_i, X_j) := \sum_{k=1}^{d} \Gamma_{ij}^{k} X_k .$$

Hence for $U = \sum_i u_i X_i$, $V = \sum_j v_j X_j$, $W = \sum_e w_e X_e$:

$$g_p(\Gamma(U,V), W) = \sum_{i,j,k,e=1}^{d} u_i v_j w_e \Gamma_{ij}^{e} g_{ke}$$

~~[illegible]~~ Inserting (**) in the ~~[illegible]~~ rhs. yields:

<u>Def 5.5</u>: (Christoffel operator)

For $p \in M$, $\Gamma = \Gamma_p : T_pM \times T_pM \to T_pM$. For $U, V \in T_pM$ we implicitly define $\Gamma_p(U,V)$ via

$$g_p(\Gamma_p(U,V), W) = \frac{1}{2}\Big((D_p g)(V)(U,W) + (D_p g)(U)(V,W) - (D_p g)(W)(U,V) \Big) \quad \forall W \in T_pM.$$

Let $\gamma : I \to M$ a curve, W a vector field along γ, that means $W(t) \in T_{\gamma(t)}M$ ~~where~~ $W(t) = \sum_{e=1}^{d} w_e(t) X_e(\gamma(t))$.

Let $\dot W(t) := \sum_e \dot w_e(t) X_e(\gamma(t))$ and $V(t) = \dot\gamma(t) = \sum_e v_e(t) X_e(\gamma(t))$.

The product rule implies:

$$\frac{d}{dt} W(t) = \sum_{e=1}^{d} \Big(\dot w_e(t) X_e(\gamma(t)) + w_e(t) \sum_{k=1}^{d} X_{,ek}(\gamma(t))\, v_k(t) \Big)$$

The covariant derivative $\frac{D}{dt} W(t)$ of W along γ is defined as the projection of $\frac{d}{dt} W(t)$ onto the tangent space.

Hence replace $X_{,ek}$ by its tangential part, i.e. $\sum_{m=1}^{d} \Gamma_{ek}^{m} X_m$ $= \Gamma_{\gamma(t)}(X_e, X_k)$

$$\frac{D}{dt} W(t) = \dot W(t) + \Gamma_{\gamma(t)}(W(t), \dot\gamma(t))$$

Def. 5.6 (Covariant derivative)

Let $\gamma : I \to M$ be curve and W vector field along γ.

$$g_{\gamma(t)}\left(\frac{D}{dt} W(t), u\right) := g_{\gamma(t)}\left(\dot{W}(t) + \Gamma_{\gamma(t)}(W(t), \dot{\gamma}(t)), u\right) \quad \forall u \in T_{\gamma(t)}M.$$

[For given $W_0 \in T_{\gamma(t_0)}M$ one can solve $\frac{D}{dt} W(t) = 0$ with $W(t_0) = W_0$ as ODE to perform parallel transport of W_0 along γ.]

Prop 5.7 (Parallel transport)

Let $\gamma : I \to M$ curve. A vector field W along γ is called parallel if $\frac{D}{dt} W(t) = 0$ for all $t \in I$. For $t_0 \in I$ and $W_0 \in T_{\gamma(t_0)}M$, there is a unique parallel vector field W with $W(t_0) = W_0$. The map

$$P_{\gamma(t_0) \to \gamma(t)} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$$
$$W_0 \mapsto W(t)$$

is a linear isomorphism.

Remark: A curve $\gamma : I \to M$ is usually defined to be a geodesic curve, if it solves the geodesic equation:

$$\frac{D}{dt} \dot{\gamma}(t) = 0 \quad \forall t \in I.$$

Thm 5.8

If $\gamma : [0,1] \to M$ is a geodesic (in the sense of Def. 5.1) connecting $\gamma(0)$ and $\gamma(1)$, then $\frac{D}{dt}\dot{\gamma}(t) = 0 \quad \forall t \in (0,1]$.

Proof: Euler-Lagrange eqs. of $\mathcal{E}$:

$$0 = \partial_\gamma \mathcal{E}[\gamma](\vartheta) = \int_0^1 (D_\gamma g_\gamma)(\vartheta)(\dot{\gamma},\dot{\gamma}) + 2 g_\gamma(\dot{\gamma},\dot{\vartheta})\, dt$$

$$= \int_0^1 (D_\gamma g_\gamma)(\vartheta)(\dot{\gamma},\dot{\gamma}) - 2(D_\gamma g_\gamma)(\dot{\gamma})(\dot{\gamma},\vartheta) - 2 g_\gamma(\ddot{\gamma},\vartheta)\, dt$$

for <u>all</u> smooth test vector fields ϑ along γ, $\vartheta(0) = \vartheta(1) = 0$.

$$\Rightarrow 0 = g_\gamma(\ddot{\gamma},\vartheta) + (D_\gamma g_\gamma)(\dot{\gamma})(\dot{\gamma},\vartheta) - \tfrac{1}{2}(D_\gamma g_\gamma)(\vartheta)(\dot{\gamma},\dot{\gamma})$$

$$= g_\gamma(\ddot{\gamma} + \Gamma_\gamma(\dot{\gamma},\dot{\gamma}),\vartheta) = g_\gamma\left(\tfrac{D}{dt}\dot{\gamma},\vartheta\right). \qquad \square$$

In particular: $g_\gamma(\dot{\gamma},\dot{\gamma}) = \text{const}$ for geodesics.

(since $\frac{d}{dt}\left(g_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t))\right) = 0$)

Def. 5.9 (Exponential map)

Let $\gamma(t) = \gamma(t,p,V)$ be the solution of $\frac{D}{dt}\dot{\gamma}(t) = 0$ with initial data $\gamma(0) = p$, $\dot{\gamma}(0) = V$. Then

$$\exp_p : T_pM \to M, \quad \exp_p(V) := \gamma(1,p,V)$$

Remark: Since $\gamma(t,p,V) = \exp_p(tV)$ we conclude that $\gamma(1,p,V)$ is well-defined if $\|V\|$ is sufficiently small. Hence actually $\exp_p : T_pM \cap B_\varepsilon(0) \to M$ for suitable $\varepsilon > 0$.

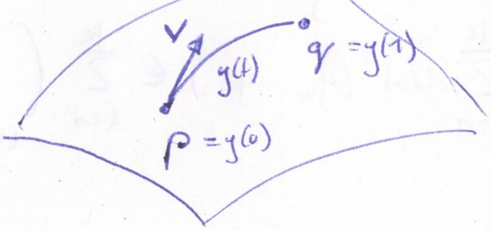
We say $\exp_p(B_\varepsilon(0))$ is the normal neighborhood of p.

<u>Def 5.10</u> (Logarithm)

The inverse operator of $\exp_p$ is called (geometric) logarithm,

i.e. $\log_p : U_p \to T_pM$, where U_p is normal neighborhood of p.

$$\exp_p(v) = q$$
$$\log_p(q) = v$$



$\exists$ unique geodesic y with

$+$ $y(0) = p$, $\dot y(0) = v$

$+$ y connects p and q (locally)!

5.2 Variational time-discretization of geodesics

Derive geodesic calculus / Riemannian structure by means of (squared) distance measure

In detail: local approximation of dist^2 given by dissimilarity measure.

Recover metric from dist^2:

$$g_p(v,w) = \frac{1}{2}\,\partial_2^2\,\mathrm{dist}^2(p,p)(v,w) \qquad , \; v,w \in T_pM.$$

<u>Notation</u>: $Y^K = (y_0, \ldots, y_K) \subset M$ ordered set of points, is denoted as <u>discrete path</u> in M, for $K \in \mathbb{N}$.

Interpretation: $y : [0,1] \to M$ curve, then $y_k = y(k\tau)$, $\tau = K^{-1}$ and $k = 0, \ldots, K$.

We have the estimates

① $L[y(t)] \geq \sum_{k=1}^{K} \mathrm{dist}(y_{k-1}, y_k)$

② $\mathcal{E}[y(t)] \geq \frac{1}{\tau} \sum_{k=1}^{K} \mathrm{dist}^2(y_{k-1}, y_k)$

<u>Proof</u>: ① ✓

② Cauchy-Schwarz:

$$\sum_{k=1}^{K} \mathrm{dist}^2(y_{k-1}, y_k) \leq \sum_{k=1}^{K} \left(\int_{(k-1)\tau}^{k\tau} \sqrt{g_y(\dot y, \dot y)}\, dt \right)^2 \underset{CS}{\leq} \sum_{k=1}^{K} \tau \int_{(k-1)\tau}^{k\tau} g_y(\dot y, \dot y)\, dt$$

$$= \tau\, \mathcal{E}[y(t)].$$

<u>Local approximation of dist^2</u>:

Assume there is $\mathcal{W} : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ s.t.

$$(*) \qquad \mathcal{W}[y, \tilde y] = \mathrm{dist}^2(y, \tilde y) + O(\mathrm{dist}^3(y, \tilde y)).$$

<u>Remarks</u>: * $\mathcal{W}$ does not have to be symmetric

* in physical shape spaces: $\mathcal{W} \triangleq$ diss. measure

* if g is given: $\mathcal{W}[y, \tilde y] := \frac{1}{2} g_y(\tilde y - y, \tilde y - y)$ is valid choice.

<u>Thm 5.11</u> (Consistency, Rumpf & Wirth '15)

Under suitable assumptions on $\mathcal{M}$ and g, and if $\mathcal{W}$ is twice differentiable, then $(*)$ implies

$\mathcal{W}[y, y] = 0$,

$\mathcal{W}_{,2}[y, y](V) = 0 \quad \forall\, V \in T_y\mathcal{M}$.

$\mathcal{W}_{,22}[y, y](V, W) = 2\, g_y(V, W) \quad \forall\, V, W \in T_y\mathcal{M}$.

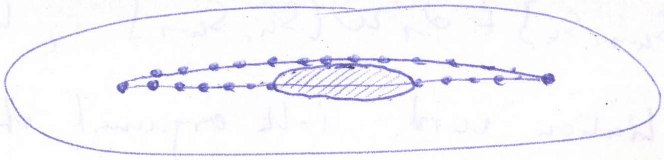
Def 5.12 (Discrete length & energy)

For $Y^K = (y_0, \dots, y_K)$ discrete path, we define

$$L^K[Y^K] = \sum_{k=1}^{K} \sqrt{\mathcal{W}[y_{k-1}, y_k]} \,, \qquad E^K[Y^K] = K \sum_{k=1}^{K} \mathcal{W}[y_{k-1}, y_k].$$

Then a discrete geodesic (of order K) connecting y_0 and y_K is defined as the minimizer of E^K for fixed end points.

Remark: Rumpf & Wirth have shown that discrete geodesics converge to continuous ones (as $E^K \xrightarrow{\Gamma} \mathcal{E}$ in suitable sense).
But: discrete minimizers of L^K are not related to continuous geodesics:



$\mathcal{W}[y, \tilde{y}] = \|y - \tilde{y}\|^2$

$\mathcal{M} = \mathbb{R}^2 \setminus B_r(0)$

Rumpf & Wirth have shown existence and (local) uniqueness of discrete geodesics.

Application to discrete shell space

Def 5.17 (Shape space of discrete shells)

Given some reference discrete shell S, the shape space of discrete shells $\mathcal{M}[S]$ is given by the equivalence class of S where the equiv. relation is given by dense correspondence.

Two implications:

1) definition of diss. measure is well-defined
2) identify $\mathcal{M} = \mathcal{M}[S]$ with $\mathbb{R}^{3n}$, where $n = |V|$

Def. 5.18 (Time-discrete geodesics in discrete shell space)

For $S_A, S_B \in \mathcal{M} = \mathbb{R}^{3n}$ we refer to the minimizer $(S_0, \dots, S_K)$ of

$$E^K[S_0, \dots, S_K] = K \sum_{k=1}^{K} \mathcal{W}[S_{k-1}, S_k]$$

with $S_0 = S_A$ and $S_K = S_B$ as (time-) discrete geodesic.

Here the diss. measure $\mathcal{W}$ has been defined in Def. 4.7.

Necessary conditions:

$$(*) \begin{cases} 0 = \partial_{S_k} E^K[S_0, \dots, S_K] & , \; k = 1, \dots, K-1 \\ \Leftrightarrow \quad 0 = \partial_2 \mathcal{W}[S_{k-1}, S_k] + \partial_1 \mathcal{W}[S_k, S_{k+1}] & , \; k = 1, \dots, K-1 \end{cases}$$

where $\partial_i \mathcal{W}$ is differentiation wrt. i-th argument of $\mathcal{W}$.

<u>Notation</u>: $0 = \partial_X F[X] \quad :\Leftrightarrow \quad 0 = \frac{d}{dt}\left(F[X+tV]\right)\Big|_{t=0} \quad \forall V$

To compute discrete geodesics we have to solve the system of nonlinear equations $(*)$ <u>simultaneously</u>! (Fixing S_0, S_K)

5.3 Time-discrete geodesic calculus

Let $p, q \in M$ s.t. there is a unique geodesic $y: [0,1] \to M$ with $y(0) = p$, $y(1) = q$. Then $\log_p(q) = \dot{y}(0) \in T_pM$.

But
$$\dot{y}(0) = \frac{y(\tau) - y(0)}{\tau} + O(\tau)$$

$$\Rightarrow \quad \tau \log_p(q) = y(\tau) - y(0) + O(\tau^2).$$

<u>Def. 5.19</u> (Discrete logarithm)

Let $Y^K = (y_0, \dots, y_K)$ be a unique minimizer of E^K in Def. 5.12, with $y_0 = p$ and $y_K = q$. Then set:

$$\left(\tfrac{1}{K} \mathrm{LOG}\right)_p (q) := y_1 - y_0$$

<u>Remark</u>: Rumpf & Wirth have shown $K \cdot \left(\tfrac{1}{K} \mathrm{LOG}_p\right)(q) \to \log_p(q)$.

<u>Cont. setup</u>: $\exp_p(V) = y(1)$ where $y: [0,1] \to M$ geodesic with $y(0) = p$ and $\dot{y}(0) = V$.

We have the scaling $\exp_p(t_k V) = y(t_k)$ for $k = 0, \dots, K$ with $t_k = k\tau$, $\tau = K^{-1}$.

Now let $Y^K = (y_0, \dots, y_K)$ discrete geodesic, $y_0 = p$, $y_K = q$.

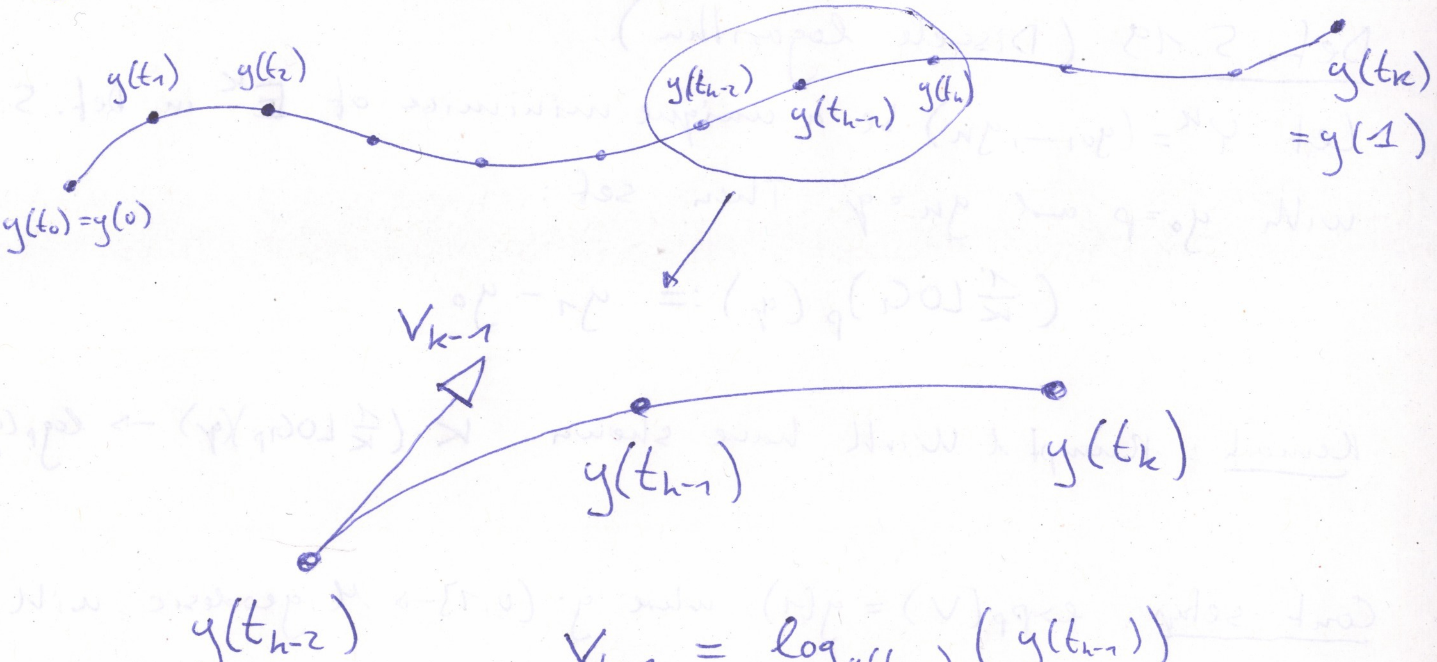
Let $V := \left(\tfrac{1}{K} \mathrm{LOG}\right)_p (q)$.

Aim at discrete exponential map EXP_p^k, s.t.

$$\mathrm{EXP}_p^k (V) := \mathrm{EXP}_p (kV) := y_k.$$

Furthermore, in the continuous setup, the following recursive relation holds:

$$\left.\begin{aligned} y(t_k) &= \exp_p(kV) = \exp_{y(t_{k-2})}(2V_{k-1}), \\ &\text{with } V_{k-1} = \log_{y(t_{k-2})} y(t_{k-1}) \end{aligned}\right\} k \geq 2.$$



$$V_{k-1} = \log_{y(t_{k-2})}(y(t_{k-1}))$$

$$\Rightarrow \exp_{y(t_{k-2})}(V_{k-1}) = y(t_{k-1})$$

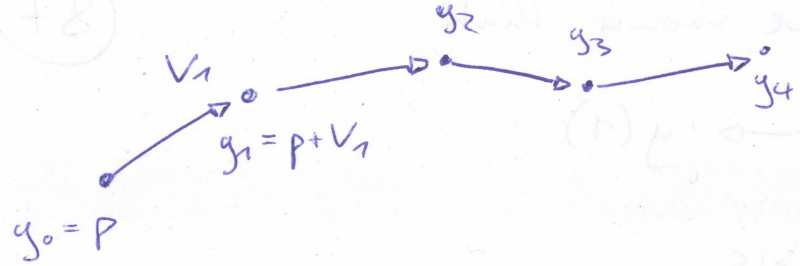
$$\Rightarrow \exp_{y(t_{k-2})}(2V_{k-1}) = y(t_k)$$

$\Rightarrow$ Given $y(t_0)$ and $y(t_1)$ one can compute $y(t_k)$ for $k \geq 2$ recursively!

Translate to discrete setup:

$$y_k = \mathrm{EXP}_p^k(V_1) = \mathrm{EXP}^2_{y_{k-2}}(V_{k-1}), \quad V_{k-1} = y_{k-1} - y_{k-2}$$

for given $y_0 = p$ and $y_1 = p + V_1$.



Remains to define EXP_p^2 operator!

Let $y_0, y_1 \in \mathcal{M}$ be given. Then we are looking for $y_2 \in \mathcal{M}$, s.t. (y_0, y_1, y_2) is discrete geodesic of order 2.

$\Rightarrow \quad EXP^2_{y_0}(y_1 - y_0) := y_2$!

Necessary condition: $\quad 0 = \partial_2 \mathcal{W}[y_0, y_1] + \partial_1 \mathcal{W}[y_1, y_2]$

Def. 5.20 (Discrete exponential map)

Given $y_0, y_1 \in \mathcal{M}$, $V_1 := y_1 - y_0$. Define $EXP^2_{y_0}(V_1)$ as solution of

$$\partial_2 \mathcal{W}[y_0, y_1](\psi) + \partial_1 \mathcal{W}[y_1, y_2](\psi) = 0 \quad \forall \psi$$

and hence $EXP^k_{y_0}(V_1) = EXP^2_{y_{k-2}}(V_{k-1})$, $V_{k-1} = y_{k-1} - y_{k-2}$, for $k \geq 2$.

Remark: If we want to compute $EXP^k_{y_0}(V_1)$ for $k = 2, \dots, K$ we have to solve the following equations sequentially:

$$0 = \partial_2 \mathcal{W}[y_0, y_1] + \partial_1 \mathcal{W}[y_1, y_2]$$
$$0 = \partial_2 \mathcal{W}[y_1, y_2] + \partial_1 \mathcal{W}[y_2, y_3]$$
$$\vdots$$
$$0 = \partial_2 \mathcal{W}[y_{k-1}, y_k] + \partial_1 \mathcal{W}[y_k, y_{k+1}]$$
$$\vdots$$

$\Rightarrow$ corresponds exactly to necessary conditions for discrete geodesic ((*) below Def. 5.18) $\rightsquigarrow$ consistency by definition!

<u>Remark</u>: Rumpf & Wirth have shown that

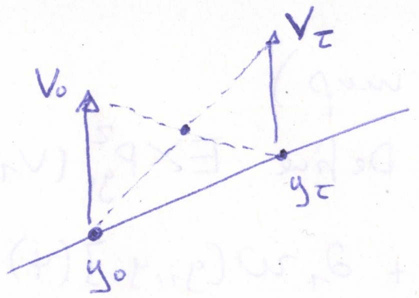
$$\mathrm{EXP}^K_{y(0)}\left(\frac{\dot y(0)}{K}\right) \longrightarrow y(1)$$

where $y:[0,1]\to\mathcal{M}$ is geodesic.

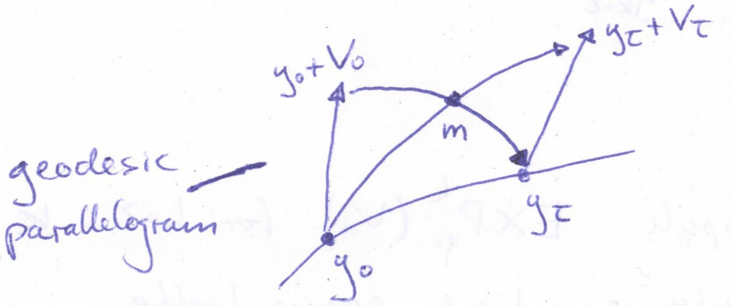
<u>Parallel transport</u>:

Given $y:[0,1]\to\mathcal{M}$ path, $V_0\in T_{y(0)}\mathcal{M}$, then parallel transport $P_{y(0)\to y(\tau)}V_0$ is defined as solution of $\frac{D}{dt}V(t)=0$ for $t\in[0,\tau]$ with $V(0)=V_0$.

<u>In $\mathbb{R}^2$</u>:



$\rightsquigarrow$ replace straight lines in $\mathbb{R}^2$ by geodesics in $\mathcal{M}$!



(Schild's ladder)

$$V_\tau \approx P_{y_0\to y_\tau}V_0$$

<u>In detail</u>:

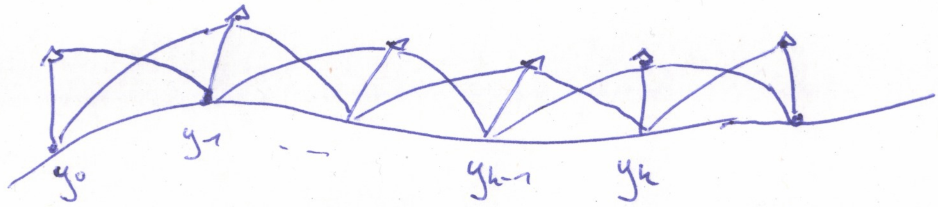
① $m=\tilde y(\tfrac12)$, if $\tilde y:[0,1]\to\mathcal{M}$ is geodesic with $\tilde y(0)=y_0+V_0$, $\tilde y(1)=y_\tau$

$$m=\exp_{y_0+V_0}\left(\tfrac12\log_{y_0+V_0}(y_\tau)\right)$$

② $y_\tau+V_\tau=\hat y(1)$, if $\hat y:[0,1]\to\mathcal{M}$ is geodesic with $\hat y(0)=y_0$, $\hat y(\tfrac12)=m$.

$$y_\tau+V_\tau=\exp_{y_0}\left(2\log_{y_0}(m)\right)$$

This procedure can be iterated!



<u>Now</u>: Translate to discrete setup!

<u>Def. 5.21</u> (Discrete parallel transport)

Let $(y_0,\dots,y_K)$ discrete path in M, ξ_0 a (sufficiently small) displacement of y_0, let $y_0^p := y_0 + \xi_0$.

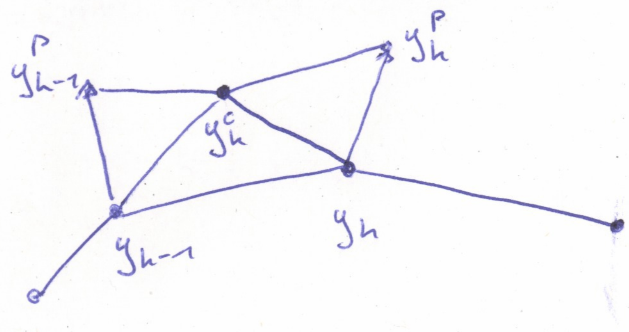
Then the discrete parallel transport of ξ_0 along y^K is defined for $k=1,\dots,K$ via the iteration

$$y_k^c := y_{k-1}^p + \left(\tfrac{1}{2}\mathrm{LOG}\right)_{y_{k-1}^p}(y_k)$$

$$y_k^p := \mathrm{EXP}^2_{y_{k-1}}\left(y_k^c - y_{k-1}\right)$$

where $\xi_k := y_k^p - y_k$ is the transported ~~displacement~~ displacement at y_k. We define

$$P_{y_K,\dots,y_0}(y_0^p - y_0) = y_K^p - y_K$$



<u>Remark</u>: Rumpf & Wirth have shown for $K \to \infty$

$$K\, P_{y_K,\dots,y_0}\left(\frac{\xi(0)}{K}\right) \to \xi(1)$$

where $\xi : [0,1] \to TM$ is a parallel vector field along $y:[0,1] \to M$.