

V4E2 - Numerical Simulation

Sommersemester 2018
Prof. Dr. J. Garcke
Teaching assistant: B. Paparella
Tutor: Marko Rajković



Exercise sheet 1.

To be handed in on **Tuesday, 24.4.2018.**

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a Hamiltonian and $\Omega \subset \mathbb{R}^d$ be an open domain. We consider the problem

$$H(x, u, Du) = 0, \quad \forall x \in \Omega.$$

and we assume H to satisfy our usual assumptions:

[A1] $H(\cdot, \cdot, \cdot)$ is uniformly continuous on $\Omega \times \mathbb{R} \times \mathbb{R}^d$

[A2] $H(x, u, \cdot)$ is convex on \mathbb{R}^d

[A3] $H(x, \cdot, p)$ is monotone on \mathbb{R} .

The definition of **viscosity solution** for this problem was given in the lecture.

Exercise 1. (A first example)

Check that

$$u(x) = \begin{cases} x & , \quad 0 < x \leq \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} < x < 1 \end{cases}$$

is a viscosity solution of $H(x, u, Du) := |u'(x)| - 1 = 0$, $x \in (0, 1)$. Is u a viscosity solution of $-|u'(x)| + 1 = 0$ in $(0, 1)$?

(4 Punkte)

Exercise 2. (H and change of sign - !!! UPDATED - there was a typo here !!!)

Check that u is a viscosity solution for $H(x, w, Dw) = 0$ iff $-u$ is a viscosity solution for $-H(x, -w, -Dw) = 0$.

(4 Punkte)

An alternative way for defining viscosity solutions is provided with the help of sub- and super-differentials. We will see this better during the lecture, but for the moment it is useful to develop some manuality with the definitions.

Definition 1 (Sub- and Super- differentials). Let Ω be an open set in \mathbb{R}^d and $v : \Omega \rightarrow \mathbb{R}$. The super-differential $D^+v(x)$ of v at $x \in \Omega$, is defined as the set

$$D^+v(x) := \left\{ p \in \mathbb{R}^d : \limsup_{\substack{y \rightarrow x \\ y \in \Omega}} \frac{v(y) - v(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}.$$

The sub-differential $D^-v(x)$ of v at $x \in \Omega$, is defined as the set:

$$D^-v(x) := \left\{ q \in \mathbb{R}^d : \liminf_{\substack{y \rightarrow x \\ y \in \Omega}} \frac{v(y) - v(x) - q \cdot (y - x)}{|y - x|} \geq 0 \right\}.$$

Exercise 3. (An example)

a) Let

$$v_1(x) := |x|.$$

Compute $D^+v_1(0)$ and $D^-v_1(0)$.

b) Let

$$v_2(x) := \begin{cases} 0 & , \quad x \leq 0 \\ \frac{1}{2}bx^2 + ax & , \quad x > 0. \end{cases}$$

Compute $D^+v_2(0)$.

(5 Punkte)

Exercise 4. (Subdifferential for convex functions)

Prove: If $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex (i.e. $u(\lambda x + (1 - \lambda)y) \leq \lambda u(x) + (1 - \lambda)u(y)$, for any $x, y \in \mathbb{R}^d$, $\lambda \in [0, 1]$), then its sub-differential at x in the sense of convex analysis is the set

$$\partial_c u(x) := \{p \in \mathbb{R}^d : u(y) \geq u(x) + p \cdot (y - x), \forall y \in \mathbb{R}^d\}.$$

Show that if u is convex then $\partial_c u(x) = D^-u(x)$.

(4 Punkte)