V4E2 - Numerical Simulation

Sommersemester 2018
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Exercise sheet 10.

To be handed in on **Tuesday, 03.07.2018**.

In this problem list we conclude the study of the Minimum Time Problem (MTP), already started in the previous exercise sheet. Consequently, **we assume to be exactly in the same setting of Problem Sheet 9, exercise 3, MTP section**.

We already know from the previous week the Dynamic Programming Principle formulation for this problem, i.e. that assuming (A_0), (A_3), one has for all $x \in \mathcal{R}$,

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{\min\{t, t_x(\alpha)\} + \chi_{\{t \le t_x(\alpha)\}} T(y_x(t, \alpha))\},$$

for all $t \ge 0$, and

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{t + T(y_x(t, \alpha))\},$$

for all $t \in [0, T(x)]$. As said, this is a result from Problem Sheet 9.

We start the new journey by checking an uniqueness property.

Exercise 1. (Uniqueness)

Let a function $S : \mathbb{R}^N \to [0, +\infty]$ satisfy the Dynamic Programming Principle (DPP) at given point $x \in \mathcal{R}$, that is

$$S(x) = \inf_{\alpha \in \mathcal{A}} \{t \wedge t_x(\alpha) + \chi_{\{t \le t_x(\alpha)\}} S(y_x(t, \alpha))\}, \quad \text{for all } t \ge 0.$$

Prove that $S(x) = T(x)$.

(4 Punkte)

Exercise 2. (HJB equation for the MTP : case with no boundary)

As usually done with many other cases, we aim to to obtain a HJB equation for this problem. Maybe a good way could be to start with the classical differentiable setting, recalling that HJB is the infinitesimal version of DPP.

Prove that if $T \in C^1$ in a neighborhood of $x \in \mathcal{R} \setminus \mathcal{T}$, then it satisfies:

$$\sup_{a \in A} \{-DT(x) \cdot f(x, a)\} = 1$$

We move to viscosity case defining the Hamiltonian:

$$H(x, p) \doteq \sup_{a \in A} \{-p \cdot f(x, a)\} - 1$$

Prove that if $\mathcal{R} \setminus \mathcal{T}$ is open and T is continuous, then T is a viscosity solution of $H(x, Dx) = 0$ on $\mathcal{R} \setminus \mathcal{T}$.

(2+4 Punkte)

Exercise 3. (Full HJB equation for MTP)

We want now to state a more complete HJB equation for the MTP setting that takes into account also the boundary conditions. Indeed, it is quite natural to think about extending T to the border requiring:

$$T(x) = 0, \quad x \in \partial \mathcal{T}$$

If it is straightforward in the classical sense, when we study the complete equations in the viscosity sense, the boundary condition must be checked in the weak form, i.e. by verifying the inequalities explained in class and summed up in the previous sheet.
For this purpose, it is generally useful to have controls capable of rotating f on the boundary in order to have a good angles, so we assume the following Small-Time-Controllable condition:

Definition 1 (STC condition)**.** The set $\partial\mathcal{T}$ is smooth and, for each $x \in \partial\mathcal{T}$, there exists a control vector $\hat{a} \in A$ such that:

$$f(x, \hat{a}) \cdot \nu(x) < 0$$

where $\nu(x)$ is the exterior normal to $\partial\mathcal{T}$ at x.

Prove that if $\mathcal{R} \setminus \mathcal{T}$ is open and T is continuous, then T is a viscosity solution of:

$$\begin{cases} H(x, Du(x)) = 0 & x \in \mathcal{R} \setminus \mathcal{T} \\ u(x) = 0 & x \in \partial\mathcal{T} \end{cases}$$

(6 Punkte)