V4E2 - Numerical Simulation

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Prof. Dr. J. Garcke
Teaching assistant: Biagio Paparella
Tutor: Marko Rajković
(marko.rajkovic@uni-bonn.de)

Exercise sheet 2. To be handed in on **02.05.2018 (during the exercise session)**.

Let $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a Hamiltonian and $\Omega \subset \mathbb{R}^d$ be an open domain. As usual, we consider the (stationary) problem:

$$H(x, u, Du) = 0, \quad \forall x \in \Omega. \tag{1}$$

with the ordinary assumptions here omitted.

Exercise 1. (Test functions are actually the usual one)

Show by a density argument that an equivalent definition of viscosity solution for (1) can be given by using $C^\infty(\Omega)$ instead of $C^1(\Omega)$ as the 'test function space'. [Hint: Friedrichs mollifier]

(4 Punkte)

Exercise 2. (Min/max and change of coordinates)

Prove *one* of the following. You can also prove both, gaining then *extra* points:

- It is false, in general, that if u, v are viscosity solutions of (1) the same is true for $u \wedge v$, $u \vee v$ (give a counterexample).
- Let $u \in C(\Omega)$ be a viscosity solution of (1), and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a function in $C^1(\mathbb{R})$ such that $\Phi'(t) > 0$. Then $v \doteq \Phi(u)$ is a viscosity solution of

  $$H(x, \Psi(v(x)), \Psi'(v(x))Dv(x)) = 0, \forall x \in \Omega$$

  where $\Psi = \Phi^{-1}$ denotes the inverse of Φ.

(3+3 Punkte)

Exercise 3. (A sufficient condition)

Let $A \subset \mathbb{R}^k$ be compact $f : \Omega \times A \to \mathbb{R}^d$ and $l : \Omega \times A \to \mathbb{R}$. Suppose that for all x, y, a

$$|f(x,a) - f(y,a)| \leq L|x - y|, \quad |\ell(x,a) - \ell(y,a)| \leq \omega(|x - y|)$$

where the constant L and the modulus ω are independent of $a \in A$. Let $H(x,p) = \sup_{a \in A}\{-f(x,a) \cdot p - \ell(x,a)\}$. Show that H satisfies:

$$|H(x,p) - H(y,p)| \leq \omega_1(|x - y|(1 + |p|))$$

($\omega_1 : [0, +\infty[\to [0, +\infty[$ is continuous nondecreasing with $\omega_1(0) = 0$).
Why is this property important?

(4 Punkte)

For this last exercise we consider the time-dependent case reformulation of equation (1), whose precise definition and notion of viscosity solution have been given in the lecture.

Exercise 4. (An example with time-dependency)

Check that both $u_1(t,x) = 0$ and $u_2(t,x) = (t - |x|)^+$ are viscosity supersolutions of

$$u_t - |u'(x)| = 0, (t,x) \in (0, +\infty) \times \mathbb{R}$$

$$u(0,x) = 0, x \in \mathbb{R}$$

Is u_2 a subsolution? [Hint: look at $D^+u_2(1,0)$]

(4 Punkte)